A suitable computational strategy for the parametric analysis of problems with multiple contact

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SUMMARY

The aim of the present work is to develop an application of the LArge Time INcrement (LATIN) approach [1] for the parametric analysis of static problems with multiple contacts. The methodology adopted was originally introduced to solve viscoplastic and large-transformation problems [2, 3]. Here, the applications concern elastic, quasi-static structural assemblies with local nonlinearities such as unilateral contact with friction. Our approach is based on a decomposition of the assembly into substructures and interfaces. The interfaces play the vital role of enabling the local nonlinearities, such as contact and friction, to be modeled easily and accurately. The problem on each substructure is solved by the finite element method and an iterative scheme based on the LATIN method is used for the global resolution. More specifically, the objective is to calculate a large number of design configurations. Each design configuration corresponds to a set of values of all the variable parameters (friction coefficients, prestress) which are introduced into the mechanical analysis. A full computation is needed for each set of parameters. Here we propose, as an alternative to carrying out these full computations, to use the capability of the LATIN method to re-utilize the solution to a given problem (for one set of parameters) in order to solve similar problems (for the other sets of parameters).

KEY WORDS: parametric uncertainties; contact; friction; nonincremental method; substructuring method; multiresolution

1. INTRODUCTION

For structural engineers, the incorporation of the system’s parametric uncertainties into a problem represents a challenge; however, without this information, the structural response could not be analyzed accurately. These parametric system uncertainties may affect the material’s mechanical properties (modulus and strength, etc.), the geometric properties (cross-sectional properties and dimensions), the boundary conditions (including contact with friction), the magnitude and distribution of loads, etc. In order to take these uncertainties into account, it is necessary to calculate the response of the structure for each set of values of the design parameters. Typically, in our case, the design parameters are the friction coefficients and the prestress. Upon completion of these calculations, response parameters (maximum stress,
reaction force...) are obtained and associated with the values of the input parameter. Then, the design space can be explored and defined by sampling parameter values within defined limits. Exploring this design space taking these parametric uncertainties into consideration is much more difficult than solving the general parametric field problem because one must take into consideration nonlinear structural behavior (in our case, unilateral contact with friction). The perturbation method [4, 5] is one of the main approaches used to obtain such structural responses. In recent years, many researchers have focused on the stochastic finite element method, which can deal with the system's parametric uncertainties mentioned above [33].

The objective of the work presented here is to develop a strategy suitable for multi-resolution problems. Therefore, the choice of an appropriate and efficient computational method is of vital importance. Some multi-resolution strategies have been proposed for optimization problems [6]. In our case, the optimized structural topology at a certain resolution level is used as an initial guess for the optimization at the next level. This provides an efficient implementation of the multi-resolution optimization technique.

The LATIN method [1] is nonincremental by nature in the sense that it accounts for the whole loading process in a single time increment which is not limited a priori. To give an idea of the step length, several loading cycles (or even several thousand cycles) can be simulated in a single time increment. This is an iterative method which differs drastically from the usual step-by-step procedures. Thus, it appears to be a promising approach, since its underlying principles tend to be more generally applicable than those of most conventional incremental algorithms. The strategy proposed here is based on the LATIN method and, more specifically, on its capability to re-utilize the solution obtained for one set of parameters in order to solve similar problems (for the other sets of parameters). This approach has already been used for viscoplastic and large-deformation problems [2, 3] and is now also being used for structural identification problems [7]. It allows one to reduce significantly the total computing cost of determining the design space.

In this paper, we consider the case of assemblies of elastic structures connected with frictional contact conditions and we limit ourselves to quasi-static loading conditions.

Contact problems are characterized by constraints such as non-penetration conditions, and an active area of contact - that is, an area where contact effectively occurs - that is unknown a priori. For these reasons, these problems lead to stiff non-linear systems of equations. Several approaches exist for solving static contact problems [8, 9, 10]. In most of them, the numerical methods that are employed for enforcing the contact constraints can be grouped into Lagrange multiplier and penalty methods [11]. The penalty methods [12, 13] are closely related to the regularization of the contact constraints. They are usually formulated in terms of the displacement variables, and therefore are primal methods. They allow treating contact as a material behavior, as exemplified by the method of joint finite elements [14]. Penalty methods can experience various numerical difficulties, especially ill-conditioning, when a too large or too small penalty parameter is introduced. Lagrange multiplier methods are dual methods where the multipliers, which represent the contact reaction forces, are introduced in order to enforce exactly the non-penetration conditions. Augmented Lagrange multiplier methods [15, 16, 17, 18] result in mixed formulations involving both displacement and force unknowns. The numerical solution schemes underlying both the Lagrange multiplier and augmented Lagrange multiplier methods are often related to the Uzawa algorithm [19, 20, 21].

Uniqueness of finite element solutions for frictional elastic problems have never been proved in the literature for general cases in elasticity. Some examples of non unique solutions have
been obtained for non analytical non smooth loading conditions on very simple systems in [22]. Uniqueness of continuum and finite element solutions have been proved for quasi-static frictional contact problems when a non local friction law is used and for small friction coefficient only [23, 24]. The friction law that is used in this paper is the classical Coulomb’s law. A finite element discretisation of the contact forces leads to a regularised non local treatment of the friction conditions. Moreover, the loss of adherence on a contact zone during the application of the external load can lead to the existence of critical points. When critical points exist, classical incremental methods must use specific continuation techniques [25, 26, 27]. As the LATIN method build an approximation of the solution on the whole time interval at each increment, it can be seen as a continuation technique when the problems contains critical points. This particular aspect on the LATIN method have successfully been applied to large displacement problems where snap throw appears [3]. The examples that are proposed in this paper do not contain any situations of critical points.

2. THE LATIN METHOD

Here, we will review only the main aspects of the LATIN method. The details of the method itself can be found in [1] and those of its particular application to computational contact problems in [28, 29].

2.1. Decomposition of an assembly

An assembly is composed of a set of substructures (each substructure is a component of the assembly) which communicate with one another through interfaces (each interface represents a connection), see Fig. 1. Each interface is a mechanical entity with its own variables and its specific behavior which depends on the type of connection. Many different connection types can be modeled by this approach, but in this paper we consider only classical contact connections. Two connected substructures are denoted $\Omega_E$ and $\Omega_{E'}$ and the associated interface is designated by $\Gamma^{EE'}$.

The interface variables are two force fields $f^E$ and $f^{E'}$ and two dual velocity fields $\dot{w}^E$ et $\dot{w}^{E'}$.

![Figure 1. Decomposition of an assembly](image-url)

(Fig. 2). By convention, $f^E$ and $f^{E'}$ are the actions of the interface on the substructures and $\dot{w}^E$ et $\dot{w}^{E'}$ are the velocities of the substructures seen from the interface.
2.2. The problem in the substructures

The displacement field at any point $M$ of $\Omega_E$ and at any time $t$ of $[0,T]$ is $u^E(M,t)$; the associated space is $\mathcal{U}^{[0,T]}$. $\epsilon$ is the strain field and the current state of the structure is characterized by the stress field $\sigma^E$.

The mechanical problem to be solved in each substructure is:

Find the histories of the displacement field $u^E(M,t)$ and stress field $\sigma^E(M,t)$ such that:

- Kinematic admissibility:
  $$\epsilon = \epsilon(u^E); \quad u^E(M,t)|_{\partial \Omega_E} = w^E(M,t); \quad u^E \in \mathcal{U}^{[0,T]}$$

- Equilibrium: $\forall \hat{u}^* \in \mathcal{U}^{[0,T]}$
  $$\int_{\Omega_E} Tr(\sigma^E(\epsilon(\hat{u}^*))) d\Omega_E - \int_{\Omega_E} f_d \hat{u}^* d\Omega - \int_{\partial\Omega_E} f^E \hat{u}^* dS = 0$$

where $\mathcal{U}^{[0,T]}$ is the set of finite-energy velocity fields on $\Omega_E$ which vanish on $\partial \Omega_E$ and $\hat{u}^*$ a virtual velocity field.

- Elastic behavior: $\forall M \in \Omega_E$ and $\forall t \in [0,T]$,
  $$\sigma^E(M,t) = D\epsilon(u^E(M,t))$$

where $D$ is Hooke’s operator.

2.3. The problem on the interfaces

The mechanical problem to be solved on each interface is:

Find the histories of the force fields $(f^E(M,t)$ and $f^{E'}(M,t))$ and of the velocity fields $(w^E(M,t)$ and $w^{E'}(M,t))$ such that:

- Equilibrium: $\forall M \in \Gamma^{EE'}$ and $\forall t \in [0,T]$,
  $$f^E(M,t) + f^{E'}(M,t) = 0$$

- Behavior: $\forall M \in \Gamma^{EE'}$ and $\forall t \in [0,T]$,
  $$f^E(M,t) = R(\dot{w}^{EE'}(M,\tau), \tau \in [0,t])$$
where the behavior is expressed as a nonlinear evolution law $\mathcal{R}$ between the forces and the rate $\dot{\mathbf{w}}^{EE'}$ of jump in displacement across the interface which is defined by:

$$\dot{\mathbf{w}}^{EE'} = \dot{\mathbf{w}}' - \dot{\mathbf{w}}^E$$

(6)

For example, a perfect connection between two substructures would be modeled by the following behavior:

$$\dot{\mathbf{w}}^{EE'}(M,t) = 0$$

(7)

The form of the evolution law $\mathcal{R}$ in the case of frictional contact conditions is described in section 2.5.

2.4. The LATIN algorithm

A LATIN (LArge Time INcrement) approach [1] is used to solve the problem. The solution $s$ of the problem is written as a set of time-dependent fields on each substructure and related interfaces:

$$s = \sum_{E} s^E ; \quad s^E = \{ u^E(M,t), \sigma^E(M,t), \dot{\mathbf{w}}^E(M,t), f^E(M,t) \} \quad t \in [0,T]$$

The LATIN approach is based on the idea of isolating the difficulties in order not to have to solve a global problem and a nonlinear problem at the same time. The equations are split into two groups with the following two sets of solutions:

- the set $\mathcal{A}_d$ of solutions $s^E$ to the linear equations related to the substructures (Eq. 1 to 3)
- the set $\Gamma$ of solutions $s^E$ to the local equations (which may be nonlinear) related to the interfaces (Eq. 4 to 5)

The search for the overall solution (i.e. the intersection of the two sets) is conducted iteratively by constructing approximate solutions $s$ which verify the two groups of equations alternatively on the complete time history. Thus, each iteration in the process is composed of two stages:

**Local stage:** for $s_n \in \mathcal{A}_d$ known, find $\mathbf{\hat{s}}$ such that:

$$\mathbf{\hat{s}} \in \Gamma \quad (interfaces) \quad (8)$$

$$\mathbf{\hat{s}} - s_n \in E^+ \quad (search \ direction) \quad (9)$$

**Global stage:** for $\mathbf{\hat{s}} \in \Gamma$ known, find $s_{n+1}$ such that:

$$s_{n+1} \in \mathcal{A}_d \quad (substructures) \quad (10)$$

$$s_{n+1} - \mathbf{\hat{s}} \in E^- \quad (search \ direction) \quad (11)$$

In our particular case of linear elastic substructures, the inner solution (in displacement $u^E(M,t)$ and in stress $\sigma^E(M,t)$) can easily be calculated from the boundary values ($\dot{\mathbf{w}}^E(M,t)$ and $f^E(M,t)$). Therefore, from here on, a solution $s$ will be represented only by the force and velocity fields on both sides of an interface.
The search directions are chosen such that the convergence of the algorithm is ensured [1]. These conjugate search directions depend on the scalar parameter $k_0$:

$$\hat{s} - s_n \in E^+ \equiv (\hat{f}^E - \hat{f}_n^{E}) = k_0(\hat{w}^E - \hat{w}_n^{E})$$  \hspace{1cm} (12)

$$s_{n+1} - \hat{s} \in E^- \equiv (\hat{f}_{n+1}^{E} - \hat{f}^{E}) = -k_0(\hat{w}_n^{E} - \hat{w}^{E})$$  \hspace{1cm} (13)

The solution of the problem does not depend on the value of the parameter $k_0$. It only affects the convergence rate of the algorithm. For quasistatic cases, which are studied here, $k_0$ is given by:

$$k_0 = \frac{F}{T L_c}$$

where $E$ is the Young’s modulus, $[0, T]$ is the studied time interval and $L_c$ the largest dimension of the structure. An error indicator is used to control the convergence of the algorithm. This indicator is an energy measure of the distance between the two solutions $s_n$ and $\hat{s}$.

2.5. The frictional contact interface

The following sections describe the form of $R$ evolution law in the case of frictional contact conditions.

2.5.1. The contact problem. The outward normal at each point of an interface is designated by $n$. The projection operator on the tangential plane $P$ is defined by:

$$Pw = w - (n.w)n$$

The non-penetrating contact conditions are:

- **Open**: if $n.\hat{w}^{EE'} > 0$ then $n.\hat{f}^E = n.\hat{f}^{E} = 0$.
- **Contact**: if $n.\hat{w}^{EE'} = 0$ then $n(\hat{f}^E + \hat{f}^{E'}) = 0$.

These conditions can be written simply as:

- **Open**: $c_n > 0$.
- **Contact**: $c_n \leq O$.

where the contact indicator $c_n$ is:

$$c_n = \frac{1}{2}n.\hat{w}^{EE'} - \frac{1}{2k_0}n.(\hat{f}^E - \hat{f}^{E})$$  \hspace{1cm} (14)

2.5.2. The friction problem. Coulomb’s friction conditions can be written as:

- **Stick**: if $\|P\hat{f}^E\| < \mu|n.\hat{f}^E|$ then $P\hat{w}^{EE'} = 0$.
- **Slip**: if $\|P\hat{f}^E\| = \mu|n.\hat{f}^E|$ then $\exists \lambda > 0$ such that $P\hat{w}^{EE'} = -\lambda\hat{f}^E$.

These conditions can be written simply as:

- **Stick**: $|g_t| \geq \mu|n.\hat{f}^E|$.
- **Slip**: $|g_t| < \mu|n.\hat{f}^E|$.

where the slip indicator $g_t$ is:

$$g_t = \frac{k_0}{2}P\hat{w}^{EE'} - \frac{1}{2}P(\hat{f}^{E'} - \hat{f}^E)$$  \hspace{1cm} (15)
2.5.3. Resolution. The resolution is carried out by projecting the solution of the previous global stage onto the contact and friction conditions following the search directions:

\[
\begin{align*}
(\tilde{F}^E - F_n^E) & = k_0 (\tilde{w}_E^E - \tilde{w}_n^E) \\
(\tilde{F}^E - F_n^E) & = k_0 (\tilde{w}_E^E - \tilde{w}_n^E)
\end{align*}
\]  

(16) (17)

The status (Open, Contact, Stick or Slip) of each point of the interface is obtained explicitly since the indicators \(c_n\) and \(g_t\) can be derived from the previous solution. An implicit time integration scheme is used to calculate the contact indicator \(c_n\) expressed in terms of the displacements (see Eq. 14):

\[
\tilde{w}_E^{E'}(t+1) = \tilde{w}_E^{E'}(t) + \Delta t \tilde{w}_E^{E'}(t+1)
\]  

(18)

The initial condition of the time integration is used to take into account the initial gap. Thus, the contact indicator is written as:

\[
c_n = \frac{1}{2k_0} \left( \tilde{n} \cdot \tilde{w}_E^{E'} - \frac{1}{2k_0} \left( \tilde{F}^E - \tilde{F}^E \right) \right)
\]  

(19)

in order to obtain using the time integration (Eq. 18) and the search directions (Eq. 16 and 17):

\[
c_n(t+1) = \frac{1}{2k_0} \left( \tilde{n} \cdot \tilde{w}_E^{E'}(t+1) - \frac{1}{2k_0} \left( \tilde{F}^E(t+1) - \tilde{F}^E(t+1) \right) \right)
\]  

\[
= \frac{1}{2k_0} \left( \tilde{n} \cdot \tilde{w}_E^{E'}(t+1) - \frac{1}{2k_0} \left( \tilde{F}^E(t+1) - \tilde{F}^E(t+1) \right) + \frac{1}{2k_0} \left( \tilde{n} \cdot \tilde{w}_E^{E'}(t) \right) \right)
\]  

\[
= \frac{1}{2k_0} \left( \tilde{n} \cdot \tilde{w}_n^{E'}(t+1) - \frac{1}{2k_0} \left( \tilde{F}^E_n(t+1) - \tilde{F}^E_n(t+1) \right) + \frac{1}{2k_0} \left( \tilde{n} \cdot \tilde{w}_n^{E'}(t) \right) \right)
\]  

Thus, the contact indicator is calculated incrementally from the known solution to the previous global stage. The slip indicator \(g_t\) (Eq. 15) is calculated explicitly using the search directions (Eq. 16 and 17):

\[
g_t = \frac{k_0}{2} P \tilde{w}_n^{E'} - \frac{1}{2} P (\tilde{f}_n^{E'} - \tilde{f}_n^{E'})
\]  

(20)

2.6. Discretization

Standard Finite Element discretization is used for the displacement field in the substructures:

\[ u = [N] \{ u \} \quad \text{and} \quad \epsilon(u) = [B] \{ u \} \]

(21)

On the interfaces, a compatible discretization is applied to the velocity fields:

\[ \tilde{w}^E = [N] \{ \tilde{w}^E \} \quad \text{and} \quad \tilde{w}^E_n = [N] \{ \tilde{w}^E_n \} \]

(22)

The search directions considered (Eq. 12 and 13) lead one to choose the same discretization for the forces and for the velocity:

\[ \tilde{F}^E = [N] \{ \tilde{F}^E \} \quad \text{and} \quad \tilde{f}_n^{E} = [N] \{ \tilde{f}_n^{E} \} \]

(23)

At the local stage, the contact equations are solved directly in terms of the nodal forces and velocities. Moreover, this representation of the forces (Eq. 23) plays a regularizing role for Coulomb’s frictional problem: the friction law becomes thus non local \([23, 24]\).
2.7. Resolution for the global stage

At the global stage, the equilibrium equation (Eq. 2), which also takes into account kinematic admissibility (Eq. 1), the behavior (Eq. 3) and the search direction (Eq. 13), becomes:

\[ \int_{\Omega_E} Tr(\epsilon(u_E^n)D(\hat{u}^*))d\Omega_E = \int_{\partial\Omega_E} (\vec{t}^E - k_0(\hat{u}_n^E - \hat{w}^E)).\hat{u}^*dS \]  

(24)

\[ \int_{\Omega_E} Tr(\epsilon(u_E^n)D(\hat{u}^*))d\Omega_E + \int_{\partial\Omega_E} k_0\hat{u}_n^E.\hat{u}^*dS = \int_{\partial\Omega_E} (\vec{t}^E + k_0\hat{w}^E).\hat{u}^*dS \]  

(25)

After discretization, Eq. 25 becomes:

\[ k_0 [h^E]\{\hat{u}(t)\} + [K^E]\{u(t)\} = [h^E]\{\{\hat{f}^E\}(t)\} + k_0\{\hat{w}^E\}(t) \]  

(26)

where

\[ [h^E] = \int_{\partial\Omega_E} [N]^t[N]ds \text{ and } [K^E] = \int_{\Omega_E} \{B\}^t D \{B\}d\Omega \]

\([K^E]\) is the classical finite element stiffness matrix of substructure \(\Omega_E\) and \([h^E]\) is the boundary term related to the interfaces.

For the resolution of the differential equation (Eq. 26), a Euler implicit time integration scheme is used. Then, the boundary terms are calculated at each time step:

\[ \{\hat{w}_n^E\} = [R]\{\hat{u}_n^E\} \]  

(27)

where \([R]\) is the restriction operator on \(\partial\Omega_E\). The forces are obtained using the search direction:

\[ \{f_n^E\} = \{\hat{f}^E\} - k_0(\hat{w}_n^E - \hat{w}^E) \]  

(28)

Finally, the algorithm can be summarize as shown in table I.

2.8. Remarks

When only static cases and perfect connections are considered, this algorithm can be obtained by other approaches such as in Lions [35] or Glowinski and Le Tallec [36]. In the case of frictional contact static cases it is not very far from augmented lagrangian algorithms as presented by Simo and Laursen [20] or Zavarise and Wriggers [37].

As augmented lagrangian algorithms the number of iterations can be elevated. But here, the operators \(K\) and \(h\) remain constant and independent of the iteration and of the contact status. Thus the cost of each iteration is low.

Moreover, it is very important to observe that the velocity and force fields on the interfaces constitute the only information needed for the subsequent iterations.

3. THE STRATEGY FOR THE PARAMETRIC STUDY

At each iteration, the LATIN method leads to an approximate solution to the problem over the whole time interval. Therefore, the trick is to reuse this approximation to find the solution
Initialize

Loop on the substructures \((E)\)
  Compute \([K^E]\) and \([h^E]\).
  Factorization.

Loop on the interfaces
  \(\{\hat{w}^E_0\} = \{\hat{w}^E_0\} = \{\hat{w}^E\} = \{0\}\)
  \(\{f^E_0\} = \{f^E_0\} = \{f^E\} = \{0\}\)

Iterate \(n = 1, 2, \ldots \) until convergence

Global stage : loop on the substructures \((E)\)
  Assemble the right hand side in equation 26
  Integrate \(\{\hat{u}(t)\}\) (eqn 26)
  Compute \(\{w^E\}\) (eqn 27)
  Compute \(\{f^E_n\}\) (eqn 28)

Local stage : loop on the interfaces \((EE')\)
  Compute \(c_n\) (eqn 19)
  Compute the normal part of \(\{\hat{w}^E\}, \{\hat{w}^{E'}\}, \{f^E\}\) and \(\{f^{E'}\}\)
    with respect to the sign of \(c_n\).
  Compute \(g_t\) (eqn 20)
  Compute the tangential part of \(\{\hat{w}^E\}, \{\hat{w}^{E'}\}, \{f^E\}\) and \(\{f^{E'}\}\)
    with respect to the value of \(\|g_t\|\).

Convergence test

Table I. Algorithm

to a problem similar to the one for which it was calculated in the first place. The multiple-solution method uses the fact that the LATIN algorithm can be initialized with any solution (usually an elastic solution) provided that it verifies the admissibility conditions. Therefore, the key to our technique is to initialize the process associated with the similar problem (the "perturbed" structure) using the results of the calculation carried out on the "initial" structure. In this manner, a first approximation of the solution to the perturbed problem with a strong mechanical content is immediately available from the start.

In this particular case of elastic structures in contact, the interfaces play a vital role: they enable one to initiate the calculation on the perturbed structures without having to save all data on the substructures as well as to search for the solution of the perturbed problem with an initial solution well-suited to the target problem. In the best-case scenario, only a few iterations are necessary: the solution to the problem is obtained at low cost. If the solutions to the "initial" and "perturbed" problems are close enough, the latter can still be derived at a significantly lower cost than by using a full calculation.

For a simple parametric study, we just change the parameters between iterations. Thus, the new computation is initialized by the solution to the previous one. The algorithm of the full parametric study is summarized in Table II. If the parameters change slowly, the two solutions are close and only a few iterations are needed to reach convergence in the new calculation.
Initialize
Loop on the substructures \((E)\)
Compute \([KE]\) and \([hE]\) and factorize.
Loop on the interfaces
\[
\{\tilde{w}_E^E\} = \{\tilde{w}_E^E\} = \{\tilde{w}_E^E\} = \{0\}
\]
\[
\{f_0^E\} = \{f_0^E\} = \{f_0^E\} = \{0\}
\]
Define limits of parameter sets
Loop \(k = 1, 2, \ldots\) number of parameter sets
Restore quantities on the interfaces
Iterate until convergence
Global stage: see table I
Local stage: see table I
Convergence test
Save interface solution for \(k^{th}\) parameter set

Table II. Algorithm for the parametric study with the LATIN Method

4. EXAMPLES

4.1. A simple example

Let us consider the simple example of a cube subjected to frictional contact against a rigid body. This \(40 \times 40 \times 40\)mm cube is subjected to friction on plane \((z = 0)\). Symmetry conditions are prescribed at planes \((x = 0)\) and \((y = 0)\). The cube is pressed against the rigid body through the application of a pressure \(P_1 = 5\text{MPa}\) on its upper side. The other two sides are successively loaded and unloaded \((P_2 = 10\text{MPa} \text{ and } P_3 = 10\text{MPa})\) in order to generate friction on the contact surface. The geometry and loading conditions are detailed in Figure 3. We consider the quasi-static, small-perturbation problem with an elastic material (Young’s modulus \(E = 130000\text{MPa}\) and Poisson’s coefficient \(\nu = 0.2\)). The mesh is composed of 100 eight-node brick elements (460 degrees of freedom).

Figure 4 shows the deformed shape of the cube (magnified 400 times) for a friction coefficient \(\mu = 0.4\) after completion of the five loading stages (application of \(P_1\), application of \(P_2\), application of \(P_3\), removal of \(P_2\) and removal of \(P_3\)). Each loading stage was completed in 10 time steps.

The problem’s parameter is the friction coefficient \(\mu\), which varies from 0 to 2 in increments of 0.1. We studied the displacements of Point \(A\) (Figure 3) in the contact plane \((z = 0)\). Figure 5 shows the paths followed by Point \(A\) for the 21 values of \(\mu\).

The parametric study was initialized with the value \(\mu = 0\). Figure 6 shows the evolution of the error indicator which increases every time the friction coefficient changes.

The cost of the computation was compared with the cost of a direct approach using ABAQUS. Figure 7 shows the cost of each of the 21 computations. One can observe that in the direct calculations using ABAQUS the computational cost increased with the friction
coefficient. In the parametric study using the LATIN method, the computational cost decreased because the results of the successive computations were very close (as shown by Figure 5). Thus, even though the first calculation cost three times as much as the direct method, the global cost of the parametric study using the LATIN method was lower by a factor of 1.5. The calculations were carried out on a HP-J5000 workstation (Proc. PA8500 440Mhz - RAM 3.5Gb).

4.2. Parametric study: a multi-parameter example

On this quasi-static academic example, the influence of friction on two different contact zones was studied. The model is presented on Fig. 8. Three square parts \( h = 50 \text{mm}, \) Young’s modulus \( E = 2.1 \times 10^8 \text{MPa}, \) Poisson coefficient \( \nu = 0.3 \) are in contact. The Coulomb friction coefficients on the contact zones are denoted \( \mu_1 \) and \( \mu_2. \) A vertical load \( F_1 (F_{1\text{max}} = 50\text{MPa}) \)

![Figure 3. Cube example: the model and loading conditions](image1)

![Figure 4. Cube example: deformed shape](image2)
A horizontal load $F_2$ ($F_{2\text{max}} = 30\text{Mpa}$) was applied to the middle part in order to make it slide until it strikes a rigid wall. The initial distance between the middle part and the rigid wall was $j = 0.04\text{mm}$.

A $25 \times 25$ finite element mesh (linear four-noded elements) was used for each part. The total number of elements is 1875 and the total number of degrees of freedom was 4056. The system’s evolution was studied over 20 time steps.

Fig. 9 shows the deformed structure during the application of the lateral load in the case $\mu_1 = 0.1$ and $\mu_2 = 0.3$. Fig. 10 shows the history of displacement and pressure in two points of the structure for the same friction coefficients. In the following parametric study, the friction coefficients $\mu_1$ and $\mu_2$ were given 13 different values between 0. and 0.6 (step 0.05) and we
Figure 7. Cube example: cost of each calculation

Figure 8. The model
calculated the reaction force on the rigid wall. The reaction force varied from zero (incomplete sliding of the middle part) to the full (1500 N) applied lateral load (no friction).

We carried out 13 different calculations following the strategy described previously (Tab. II). Each used a fixed value of $\mu_1$ while the value of $\mu_2$ was modified during the iterations. Fig. 11 shows the evolution of the error indicator during the process. Each change of $\mu_2$ increased the error indicator, but the number of iterations needed decreased. For the highest values of $\mu_2$, the solutions were so close that only one or two iterations were needed to obtain a new value of the friction coefficient.

Table III shows the computation cost for each value of $\mu_2$. The calculations for each value
of $\mu_1$ can be carried out concurrently on different processors. Table IV shows a comparison of the costs of different types of calculations: a direct approach (169 calculations for each couple ($\mu_1, \mu_2$)), sequential parametric approach (13 calculations for each value of $\mu_1$ with $\mu_2$ modified during the iterations) and parallel parametric approach (13 concurrent calculations). The calculations were carried out on a PC (Proc. AMD Athlon(tm) 1.GHz - RAM 512Mb).

\begin{table}[h]
\centering
\begin{tabular}{cccccccccccc}
\hline
\ mu_2 & 0.00 & 0.05 & 0.10 & 0.15 & 0.20 & 0.25 & 0.30 & 0.35 & 0.40 & 0.45 & 0.50 & 0.55 & 0.60 \\
\hline
\ Nb \ ite & 236 & 183 & 130 & 79 & 70 & 40 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
\ Time \ (s) & 136 & 106 & 65 & 46 & 40 & 23 & 1.2 & 0.6 & 0.6 & 0.6 & 0.6 & 0.6 & 0.6 \\
\hline
\end{tabular}
\caption{Cost of the parametric calculation}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{lll}
\hline
\ Calculation & Cost \ (s) & Cost \ (h) \\
\hline
LATIN direct & 22980 & 6.4 \\
LATIN parametric & 6060 & 1.7 \\
LATIN parametric \ (parallel) & 466 & 0.13 \\
\hline
\end{tabular}
\caption{Cost comparison for different calculations}
\end{table}

In order to apply the Response Surface Methodology (RSM), we plotted the response surface from the previous results describing the evolution of the reaction force on the rigid wall with respect to the friction coefficients $\mu_1$ and $\mu_2$ (Fig. 12). The RSM was developed initially by Veneziano et al. [30]. It has become a widely-accepted procedure in structural reliability analysis [31]. Schueller et al. [32] used the RSM to model the actual limit state function of
large structures subject to static and dynamic loading.

The influence of a particular perturbation on the value of the reaction force can be examined by using the response surface (Fig. 12). With these results, one can carry out a Monte-Carlo simulation using the response surface to determine the probability that the reaction force is greater than a given value. The results are plotted on Fig. 13.

![Figure 12. Response surface of the reaction force on the rigid wall](image)

In the first case, we assumed a Gaussian distribution of both friction coefficients $\mu_1$ and $\mu_2$. The second curve is based on a uniform random distribution of coefficients $\mu_1$ and $\mu_2$. For both cases, ten thousand draws were carried out to obtain the probability that the reaction force exceed a given value as a function of this given value.

4.3. Parametric study :example of an assembly

This example deals with a bidimensional model of the section of a quadrupole prototype that was presented in [28].

Two magnetic channels are incorporated into a single iron yoke and cryostat and cooled with a helium superfluid in order to attain the very high guide field required (Fig. 14). The magnet coils tend to be long (approximately 14 meters) with an inner diameter of 56 mm; they are made of copper-clad Niobium-Titanium cables. The insulation is composed of two layers of Kapton.

The electromagnetic forces in the quadrupoles are contained by the collar structure. The collars are made of 2 mm thick austenitic steel laminations (Fig. 15). Every other pair of half-collars is turned 90 degrees and completed by two separate pole pieces. The keying and prestressing of the assembly is achieved by eight lines of wedge-shaped stainless steel keys which are progressively inserted into the grooves on the outside of the collars (Fig. 16). Once the coil-collar assemblies have been prepared and the coil interconnections completed, the yoke is positioned around the two units. In order to stiffen the entire yoke assembly, a so-called inertia
Figure 13. Probabilistic study

Figure 14. Quadrupole cross section
tube is placed around the yoke. This tube also serves as the helium vessel.

We only studied one cross section of a quadrupole in which we modeled the bobbin and two collars. The two collars were in the same plane. The relative motion of the collars was such that the preload was applied not only on the sides of the bobbin but also on its external radius.

Figure 15. View of the collars

Figure 16. Model of the quadrupole

The mechanical structure of the quadrupoles was designed both to withstand the strong forces generated in the magnet and to limit, as far as possible, coil deformations over the entire operating range. Therefore, the materials used for the most highly-stressed components must have a high load-bearing capacity, a high elastic modulus, a good level of fatigue endurance and
good behavioral characteristics at cryogenic temperatures as low as $1.9\,\text{K}$. Whenever current flows in the cables, the coils must be under compressive stress in order to avoid the occurrence of sudden cracks or movements. Coil displacements and deformations must be as small as possible. In order to satisfy these conditions, the relative dimensions of the structural components, the choice of materials and the level of prestressing must all be determined carefully. Furthermore, the peak compressive stress in the coils at room temperature must be minimized in order to prevent creeping both in the insulation and in the copper. Our main objective was to optimize the level of azimuthal prestress in order to avoid the occurrence of coil movements during the operation of the magnet. Thus, it was necessary to take frictional contact into account during the following three successive loading sequences (loading steps):

- preload due to the introduction of the keys: the introduction of the keys was not modeled completely. The keys enter the collars at the beginning of the step with a prescribed penetration which decreases to zero during the step.
- cooling down to the working temperature, i.e. from 293K to 1.9K. The material properties (Young’s modulus and thermal expansion coefficient) were assumed to be temperature-dependent.
- application of the magnetic field: a Lorentz body force field was applied to the coils. This field had been determined in a previous calculation.

The influence of the preload (angular gap $\alpha$ between the keys and the collars) and of the friction (coefficient $\mu$) on the lateral load ($N$) of the bobbin was studied. The mesh used (8424 elements, 5624 nodes and 11248 degrees of freedom) is shown on Fig. 17. Five time steps were used for each loading increment.

16 values of the preload parameter of the keys (angular gap $\alpha$) and 19 values of the friction coefficient ($\mu$) were considered. This means that 304 different calculations had to be carried out for the parametric study. These were divided into 16 different sessions (one for each value of $\alpha$) which were carried out sequentially or concurrently. Fig. 19 shows the evolution of the error indicator during the three loading steps of one of the parametric studies ($\alpha = 0.3$). A great reduction of the number of iterations (with respect to the first calculation) was obtained for the last values of $\mu$. Fig. 19 also shows the evolution of the lateral load $N$ on the bobbin during the loading steps. This load is the result of the introduction of the keys and decreases during the cooling and the application of the magnetic field.

Table V shows a comparison of the computational costs of the different strategies:
Figure 18. Von Mises’ equivalent stress in the bobbin and in the back collar

Figure 19. Evolution of the error indicator during the iteration of the multi-resolution process: preload step (a), cooling step (b) and magnetic field step (c). Evolution of the lateral load $N$ during the three steps (d) for one case ($\mu = 0.2$ and $\alpha = 0.3$).
• the 304 different calculations carried out on a single processor (direct LATIN).
• the 16 parametric strategies (one for each value of \(\alpha\)) carried out sequentially on one processor.
• the same 16 parametric strategies carried out concurrently on 16 processors.

The calculations were carried out on a PC (Proc. AMD Athlon(tm) 1.6GHz - RAM 512Mb).

Using the previous results, we plotted the response surface describing the evolution of the lateral load \(N\) in terms of the friction coefficient \(\mu\) and the angular gap \(\alpha\) (Fig. 20). This surface shows that the response is strongly influenced by the angular gap.

![Figure 20. Response surface for the section of the quadrupole](image)

Using the response surface, one can examine the influence of a particular perturbation on the value of the load \(N\). A Monte-Carlo simulation was carried out to determine the probability that the load \(N\) exceed a given value (Fig. 21).

In the first case, we assumed Gaussian distributions for both the friction coefficient \(\mu\) and the angular gap \(\alpha\). The second curve is based on a uniform random distribution of these two parameters. In both cases, ten thousand draws were carried out to obtain the probability curve.

These results follow the same trend as those presented in [34] where buckling and viscoelastic problems were considered. The use of the LATIN method combined with the RSM technique.

<table>
<thead>
<tr>
<th>Calculation</th>
<th>Cost (s)</th>
<th>Cost (h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Direct LATIN</td>
<td>103,660</td>
<td>29</td>
</tr>
<tr>
<td>Parametric LATIN (sequential)</td>
<td>34,480</td>
<td>9.5</td>
</tr>
<tr>
<td>Parametric LATIN (parallel)</td>
<td>2,155</td>
<td>0.6</td>
</tr>
</tbody>
</table>

Table V. Comparison of the costs of different calculations
enabled us to calculate probabilities very effectively: the LATIN method generated the response surface inexpensively and the RSM technique combined with Monte Carlo calculation allowed us to estimate the probability that the middle part strike the rigid wall with a given force (static example) or that the lateral force $N$ exceed a given force (assembly example).

5. CONCLUSIONS

The strategy proposed here is based on the LATIN method and, more specifically, on its capability to reuse the solution to a given problem in order to solve similar problems. Initial numerical examples showed the very good behavior of the algorithm applied to the case of multiple resolutions in the analysis of static and dynamic problems with contact and friction. The solution to the initial problem is a very good starting point for the calculations conducted on perturbed problems provided that these calculations do not exert excessive perturbations on the response. Moreover, as has already been explained, the interfaces play a vital role in allowing a considerable reduction of the computation costs. This first study was carried out on small academic problems; the next step will be to apply this strategy to large 3D structures, taking into account variations of the friction coefficient and also of the geometry and the material properties. Finally, this approach is quite general by nature and should
be applicable to a number of other nonlinear problems which require multiple solutions.

REFERENCES