Saint-Venant multi-surface plasticity model in strain space and in stress resultants

J.B. Colliat, A. Ibrahimbegovic, L. Davenne

Ecole Normale Supérieure de Cachan
Laboratoire de Mécanique et Technologie (LMT)
61 avenue du Président Wilson, 94235 Cachan cedex, France

Abstract:
In this work we present a new constitutive model for capturing inelastic behavior of brittle materials, with damage in tension. The multi-surface plasticity theory is employed to describe the damage-induced mechanisms. An original feature in that respect concerns the multi-surface criterion which limits the principal values of elastic strains, which is equivalent to Saint-Venant plasticity model. The latter allows to represent the damage both in tension and in compression. The model is further recast in terms of stress resultants and employed within a flat shell element in order to provide a very efficient tool for analysis of cellular structures.

keywords: multi-surface plasticity, stress resultant space models, shell elements, cellular structures

1 Introduction

With the present increase of the computational power, one can seek to provide a more realistic representation of the physical nature of damage mechanisms. The multiscale analysis (e.g see [MAR04]) thus becomes a very interesting option for combining the large structural model with a more detailed (small) representation. In [IBR05] we have described how to build a multiscale model for cellular structures, such as masonry wall built of hollow bricks, submitted to sustained duration of high temperature. The flat shell elements are used for modeling different panel barriers which allows to construct a detailed representation of each brick. With the model of this kind, we get a possibility of refined representation of local damage mechanisms in each block, but the number of degrees of freedom increases very quickly. The key challenge is thus to provide a very robust implementation of the local constitutive model.

One such model is proposed in this work, within the framework of multi-surface plasticity, with the goal to describe simple fracture mechanism under maximum stress value, similar to Rankine criterion (e.g see [PEA97]). The original feature of the model is to relate damage to maximum value of elastic strain, such as for St-Venant plasticity model [STV55]. The latter allows to take into account both cracking in tension, perpendicular to loading, and in compression, parallel to the loading direction, as observed in simple traction and compression tests. This difference is also accounted for when further describing the softening phenomena; namely, both the fracture energy in tension and in compression can be introduced in the list of model parameters and thus provide a more reliable description of fracture mechanisms. A careful consideration of the implementation details ensures a very robust performance of the proposed model in numerical computations.

The outline of the paper is as follows. In the next section we provide the main ingredients of the proposed multi-surface plasticity model developed within the standard thermodynamics framework.
A special care in that sense is dedicated to elaborating loading/unloading conditions, which remain quite subtle for a multi-surface plasticity model. In Section 3 we provide a generalization of this multi-surface plasticity model applicable in the framework of plates and shells, which is cast directly in terms of stress resultants. A detailed description of the numerical implementation is given in Section 4. In Section 5, we provide numerical results for several numerical simulations carried out to illustrate different model features. Concluding remarks are stated in Section 6.

2 Continuum formulation

In this section we first present the mechanical part of the model within the two-dimensional thermodynamics framework. The main novelty concerns the choice of elastic domain, which is written in the context of multi-surface plasticity. We then show the results for a couple of simple numerical examples illustrating the most important features of this model, before turning to its implementation in term of stress resultants.

2.1 Thermodynamics framework

In order to retrieve the most important feature of brittle materials like clay, that is to say a different behavior in tension and compression cases, the two-dimensional model shown here is set as the Saint-Venant model. The latter is very much alike the Rankine model, but with the key difference to define the elastic limit with respect to the principal mechanical strains, which corresponds to the experimentally observed behavior of such materials. From the thermodynamics point of view, the free energy is written,

\[ \psi(\varepsilon, \varepsilon^p, \xi) = \frac{1}{2} (\varepsilon - \varepsilon^p) : C : (\varepsilon - \varepsilon^p) + \mathcal{H}(\xi, \varepsilon - \varepsilon^p) \]  

(1)

where \( \varepsilon^e = \varepsilon - \varepsilon^p \) is the elastic strain defined in accordance with the classical decomposition of the total strain excluding thermal effects and \( \xi \) the hardening/softening variable for the isotropic case. In (1) above, a coupling between the strain/stress state and hardening/softening process is introduced in the expression of the corresponding energy \( \mathcal{H}(\xi, \varepsilon - \varepsilon^p) \). This explicit expression for hardening potentials will be given according to the softening law in (10), which allows to enter explicitly the desired value of fracture energy either in tension or compression.

According to this expression for strain energy, the state laws are:

\[
\sigma = \frac{\partial \psi}{\partial (\varepsilon - \varepsilon^p)} = C : (\varepsilon - \varepsilon^p) + \frac{\partial \mathcal{H}}{\partial (\varepsilon - \varepsilon^p)} \\
q = -\frac{\partial \psi}{\partial \xi} = -k(\xi, \varepsilon - \varepsilon^p) \quad \Rightarrow \quad \dot{q} = -D : \dot{\xi} - U : (\varepsilon - \varepsilon^p)
\]

(2b)

where \( D = \frac{\partial k}{\partial \xi} \) is the classical hardening modulus and \( U = \frac{\partial k}{\varepsilon - \varepsilon^p} \) is a third order tensor.

Invoking the principle of maximum plastic dissipation (see [LUB90]),

\[
D^p = \sigma : \varepsilon^p + q : \dot{\xi} \geq 0
\]

(3)

\[\begin{align*}
Max|_{\Phi_i = 0} \{D^p\} & \Leftrightarrow Max|_{\sigma} \\ Min|_{\sigma} \{\mathcal{L}^p\} \\
\partial \mathcal{L}^p/\partial \sigma & = -\partial D^p/\partial \sigma + \sum_{i=1}^{m} \gamma_i \partial \Phi_i/\partial \sigma = 0 \\
\partial \mathcal{L}^p/\partial q & = -\partial D^p/\partial q + \sum_{i=1}^{m} \gamma_i \partial \Phi_i/\partial q = 0
\end{align*}\]

(4)
where,
\[ L^p(\sigma, q; \dot{\gamma}) = -D^p(\sigma, q) + \sum_{i=1}^{m} \dot{\gamma}^i \Phi_i(\sigma, q) \]  

one can further obtain the flow rules,
\[ \dot{\varepsilon}^p = \sum_{i=1}^{m} \dot{\gamma}^i \frac{\partial \Phi_i}{\partial \sigma} \text{ with } \Phi_i = 0 \]  
\[ \dot{\xi} = \sum_{i=1}^{m} \dot{\gamma}^i \frac{\partial \Phi_i}{\partial q} \text{ with } \Phi_i = 0 \]

where the elastic domain is defined by a set of \( m \geq 1 \) yield functions, \( \Phi_i(\sigma, q) < 0 \) intersecting in a possibly non-smooth fashion.

### 2.2 Elastic domain

We note with respect to the last result that in the present two dimensional case, the elastic domain is defined by two independent surfaces \( \Phi_i(\xi, \varepsilon^p, \xi) < 0 \), \( 1 \leq i \leq 2 \) which are defined directly in the space of state variables \( \xi, \varepsilon^p \) and \( \xi \). An alternative representation can be given considering the spectral decomposition of strain tensor \( \varepsilon^e = \sum_{k=1}^{II} \varepsilon_k n_k \otimes n_k \) with \( \varepsilon_k \geq \varepsilon_I \), where the two surfaces are expressed as:

\[ \Phi_1(\xi, \varepsilon^e, \xi) := 2 \left( K + \frac{G}{3} \right) (\varepsilon_I - \varepsilon_y) - k(\xi) \leq 0 \]  
\[ \Phi_2(\xi, \varepsilon^e, \xi) := 2 \left( K + \frac{G}{3} \right) (\varepsilon_{II} - \varepsilon_y) - k(\xi) \leq 0 \]

The plastic deformation occurs with respect to the criterion which is expressed in principal mechanical strains. This criterion is usually referred to as Saint-Venant criterion [STV55]. By defining the criterion in strain space, we can provide the main advantage of this model in its ability to reproduce failure for both tension and compression stress states.

One can also recover the standard format of the yield criterion in the stress space, which is more efficient in numerical implementation. Namely, by employing the state laws in (2), we write the two surfaces in the stress space \( \Phi_i(\sigma, q) = \Phi_i(\sigma, \varepsilon^p, \xi) \) according to,

\[ \Phi_1(\sigma, q) := \frac{K + 4G/3}{2G} \sigma_I - \frac{K - 2G/3}{2G} \sigma_{II} - (\sigma_y - q) \leq 0 \]  
\[ \Phi_2(\sigma, q) := -\frac{K - 2G/3}{2G} \sigma_I + \frac{K + 4G/3}{2G} \sigma_{II} - (\sigma_y - q) \leq 0 \]

In Figure 1 we show the principal axis representation of this two dimensional criterion in the case of perfect plasticity. The two surfaces are therefore simply represented by straight lines. We should note that \( \Phi_1 \geq \Phi_2 \), so that the second surface can never be the only one active (see Figure 1). In pure tension mode, the limit of the elastic domain is,

\[ \Phi_1(\sigma_I) := \frac{2G}{K + 4G/3} \sigma_I - \sigma_y \leq 0 \]

and, in pure compression:

\[ \Phi_1(\sigma_{II}) := -\frac{2G}{K - 4G/3} \sigma_{II} - \sigma_y \leq 0 \]
The unsymmetrical feature of such material is so reproduced with a very few number of parameters (only three in perfect plasticity case). The only loading case which is not limited is the bi-axial compression along the two axes such as produced by hydrostatic pressure.

\[ \Phi_1 = 0 \]
\[ \Phi_2 = 0 \]

![Figure 1: Elastic domain in principal stress space - 2D case](image)

### 2.3 Hardening law

In this approach, the hardening is supposed to be isotropic, which allows us to use only a single variable with: \( \xi = \xi_1 \), and the associated flux \( q = q_{\parallel} \). The latter leads to the second state law in a simplified form:

\[ q = -k(\xi, \varepsilon^e) \]  
\[ q = -\sigma_y \left( 1 - \exp \left( -\frac{\sigma_y}{G_f(\varepsilon^e)} \xi \right) \right) \]  

We indicate in (12) that the fracture energy is supposed to change continuously from a specified value in tension \( G_t \) to compression \( G_c \) (Figure 2) according to:

\[ G_f = \frac{G_c + G_t}{2} - \frac{G_c - G_t}{2} \tanh(\beta \varepsilon^e) \]  

where \( \beta \) is a parameter to be chosen to set a more or less rapid transition.

From the physical point of view, this assumption on imposing different values for \( G_t \) and \( G_c \) is related to the meaning of the fracture energy, as the amount of energy needed to create a crack per square meter. Namely, even if the two failure modes in tension and compression stress states are captured with the same failure mechanism pertaining to the principal tensile strain, the number of cracks created in those two cases is completely different. This is illustrated in Figure 3 representing the crack pattern in simple tension and the one in simple compression test, leading to quite different dissipated energy.
2.4 Loading/unloading conditions and geometrical interpretation

Considering multi-surface plasticity case defined by a set of admissible surfaces \( \{ \Phi_i = 0 \} \), it has been shown by Simo et al. [SIM88] that the loading conditions in strain space will take the form: Given any \( \dot{\varepsilon} \),

\[
\begin{align*}
\text{if } \forall i, \quad \frac{\partial \Phi_i}{\partial \sigma} : C : \dot{\varepsilon} \leq 0 \quad & \Rightarrow \quad \dot{\varepsilon}^p = 0, \dot{q} = 0 \\
\text{if } \exists i, \quad \frac{\partial \Phi_i}{\partial \sigma} : C : \dot{\varepsilon} > 0 \quad & \Rightarrow \quad \dot{\varepsilon}^p \neq 0, \dot{q} \neq 0
\end{align*}
\] (14a)

\[
\begin{align*}
\text{if } \forall i, \quad \frac{\partial \Phi_i}{\partial \sigma} : C : \dot{\varepsilon} > 0 \quad & \Rightarrow \quad \dot{\varepsilon}^p \\ & \text{does not necessarily imply that the} \\
\text{surface remains active. In the opposite, one could finally have an active surface } i \text{ for which initial} \\
\text{guess was not showing it, i.e } \frac{\partial \Phi_i}{\partial \sigma} : C : \dot{\varepsilon} < 0.
\end{align*}
\] (14b)

According to the usual terminology of computational plasticity in application to multi-surface plasticity context, a surface is said to be active in contributing to internal variable evolution if \( \gamma^i > 0 \). Consequently, given any \( \dot{\varepsilon} \), we can show that:

\[ \dot{\gamma}^i > 0 \Leftrightarrow \frac{\partial \Phi_i}{\partial \sigma} : C : \dot{\varepsilon} > 0 \quad 1 \leq i \leq m \] (15)

In other words, for a particular surface \( i \), \( \frac{\partial \Phi_i}{\partial \sigma} : C : \dot{\varepsilon} > 0 \) does not necessarily imply that the surface remains active. In the opposite, one could finally have an active surface \( i \) for which initial guess was not showing it, i.e \( \frac{\partial \Phi_i}{\partial \sigma} : C : \dot{\varepsilon} < 0 \).

Consequently, the key point of the methodology in multi-surface plasticity context consists in determining the set of all active surfaces (e.g. see Hofstetter et al. [HOF93] for CAP model and Pearce and Bičanić [PEA97] for Rankine plasticity criterion) and then solving all the plastic admissibility constraint equations simultaneously.

In order to illustrate more clearly this main point related to the model proposed herein, we consider in detail the loading conditions (14) in the case of perfect plasticity.
In strain space, we first choose the curvilinear system basis defined by,
\[ g_i = \frac{\partial \Phi_i}{\partial \sigma} \quad i \in [1..2] \]
(16)
equipped with the scalar product defined by \( C \). Clearly this basis \( \{g_i\} \) is not an orthogonal one according to this scalar product, thus leading to distinguish between covariant and contravariant components and introduce the metric tensor \( g_{ij} \) written as:
\[ g_{ij} = (g_i, g_j) = g_i : C : g_j \]
(17)
According to the first flow rule (6a), by assuming a plastic step departing from the yield surface which would not change the elastic deformation for the case of perfect plasticity,
\[ \dot{\varepsilon} = \dot{\varepsilon}^p := \sum_i \dot{\gamma}^i g_i \]
(18)
showing that the plastic multipliers \( \dot{\gamma}^i \) can be seen as the contravariant components of \( \dot{\varepsilon} \) in the basis \( \{g_i\} \). With respect to this interpretation, we denote as,
\[ \Gamma^+ = \left\{ \dot{\varepsilon} \in S^3, \dot{\varepsilon}^p = \sum_i \dot{\gamma}^i \frac{\partial \Phi_i}{\partial \sigma} \mid \dot{\gamma}^i > 0, \forall i \right\} \]
(19)
the cone for which all plastic multipliers are strictly positive.

On the other hand, the consistency condition can be written as,
\[ \dot{\Phi}_i = \frac{\partial \Phi_i}{\partial \sigma} : \dot{\varepsilon} = \frac{\partial \Phi_i}{\partial \sigma} : C : \dot{\varepsilon} = \sum_j \dot{\gamma}^j g_{ij} : C : g_i = 0 \]
(20)
or:
\[ \frac{\partial \Phi_i}{\partial \sigma} : C : \dot{\varepsilon} = \sum_j \dot{\gamma}^j g_{ij} = \dot{\gamma}^i \]
(21)
According to this last result, the \( \frac{\partial \Phi_i}{\partial \sigma} : C : \dot{\varepsilon} \) can be interpreted as the covariant components of \( \dot{\varepsilon} \) in the basis \( \{g_i\} \). Consequently, we also denote the cone \( M^+ \) as:
\[ M^+ = \left\{ \dot{\varepsilon} \in S^3 \mid \forall i, \frac{\partial \Phi_i}{\partial \sigma} : C : \dot{\varepsilon} > 0 \right\} \]
(22)
We note in passing that the complementary cone \( M^- \) implies the elastic loading, which can be written as:
\[ M^- = \left\{ \dot{\varepsilon} \in S^3 \mid \forall i, \frac{\partial \Phi_i}{\partial \sigma} : C : \dot{\varepsilon} \leq 0 \right\} \]
(23)
According to these definitions, it should be noted that the result \( \dot{\gamma}^i > 0 \iff \frac{\partial \Phi_i}{\partial \sigma} : C : \dot{\varepsilon} > 0 \) can be geometrically interpreted as the non-matching of \( M^+ \) and \( \Gamma^+ \); see Figure 4 for general
case and Figure 5 for the present model. The particular feature of the proposed model which deserves a special care in handling concerns the fact that the second surface $\Phi_2$ could be active, while $\frac{\partial \Phi_2}{\partial \sigma} : C : \dot{\varepsilon} < 0$.

More precisely, the basis vector of the proposed model, can be written as:

$$g_1 = \frac{\partial \Phi_1}{\partial \sigma} = \frac{K + 4G/3}{2G} \mathbb{I}_1 \otimes \mathbb{I}_1 - \frac{K - 2G/3}{2G} \mathbb{I}_2 \otimes \mathbb{I}_2 \quad (24a)$$

$$g_2 = -\frac{K - 2G/3}{2G} \mathbb{I}_1 \otimes \mathbb{I}_1 + \frac{K + 4G/3}{2G} \mathbb{I}_2 \otimes \mathbb{I}_2 \quad (24b)$$

One can see from Figure (5) shows that the cone $\Gamma^+$ contains the cone $M^+$. Consequently, all the strain states residing in an area denoted by (A) in Figure (5) would lead to $\dot{\gamma}^2 > 0$, even if $\frac{\partial \sigma}{\partial \sigma} : C : \dot{\varepsilon} \leq 0$. For that reason, when considering the numerical implementation, three different cases have to be taken into account. In either the first or the second one, the trial elastic state is directly furnishing the set of active surfaces (see Figure (5)). For the last case, where the trial state belongs to the area denoted by (A), both surfaces are active and we have to compute the corresponding values of both plastic multipliers.
3 Stress resultant multi-surface plasticity model

The objective is here to show how the two dimensional case of the multi-surface model can be incorporated into a flat shell element which proved to be very useful in constructing the models for cellular structures (e.g. see [IBR05]). The shell element we use in this work combines the discrete Kirchhoff quadrilateral plate with a membrane element including the drilling degrees of freedom (see [IBR90] and [IBR93b]).

We denote here as $g(x_1, x_2)$ the in-plane membrane strain tensor and $k(x_1, x_2)$ the curvature tensor which can be written in a condensed form of a set of generalized strains as $\chi = (\mathbf{C})$. Normal membrane forces tensor $N$ and bending moment tensor $M$ are also written in a condensed form as $\pi = (N, M)$. We can then write the constitutive tensor for stress resultant form as,

$$\Xi = [h C : 0 : \frac{h^3}{4}] : \mathbf{C} = (\mathbf{K} - 2G/3) \mathbf{I} + 2G \mathbf{I}$$

where, $h$ is the shell thickness.

The last is used to recast in the stress resultant form the standard thermodynamical framework with the free energy,

$$\psi (\chi, \chi^p, \xi) = \frac{1}{2} (\chi - \chi^p) : \Xi : (\chi - \chi^p) + H(\xi, \chi - \chi^p)$$

which provides the state laws given as,

$$\pi = \frac{\partial \psi}{\partial (\chi - \chi^p)} = \Xi : (\chi - \chi^p) + \frac{\partial H}{\partial (\chi - \chi^p)}$$

$$\dot{q} = -\frac{\partial \psi}{\partial \xi} = -k(\xi, \chi - \chi^p) \quad \dot{\xi} = -D \cdot \dot{\xi} - U : (\chi - \chi^p)$$

where $D = \frac{\partial k}{\partial \xi}$ and $U = \frac{\partial k}{\chi - \chi^p}$ have the same meaning as in (2b). In the computation of stress resultants, the second term of (18a) is neglected.

By assuming that the same constitutive equations remain valid in the case of inelastic process, we can obtain the plastic dissipation as:

$$D^p = \pi : (\chi - \chi^p) + q \cdot \dot{\xi} \geq 0$$

The elastic domain can be defined directly in terms of stress resultants by a set of $m \geq 1$ yield functions $\Phi_i(\pi, q)$, intersecting in a possibly non-smooth fashion. With such a model ingredient, the flow rules can be written by invoking the principle of maximum plastic dissipation as:

$$Max \{ D^p \} \Rightarrow Max \{ L^p \} \equiv \left\{ \begin{array}{l}
\frac{\partial L^p}{\partial \pi} = -\frac{\partial D^p}{\partial \pi} + \sum_{i=1}^{m} \dot{\gamma} \frac{\partial \Phi_i}{\partial \pi} = 0 \\
\frac{\partial L^p}{\partial q} = -\frac{\partial D^p}{\partial q} + \sum_{i=1}^{m} \dot{\gamma} \frac{\partial \Phi_i}{\partial q} = 0
\end{array} \right.$$  (29)

where,

$$L^p(\pi, q, \dot{\gamma}) = -D^p(\pi, q) + \sum_{i=1}^{m} \dot{\gamma} \Phi_i(\pi, q)$$  (30)

and then leading to the flow rules:
\[\dot{\chi}^p = \sum_{i=1}^{m} \gamma^i \frac{\partial \Phi_i}{\partial \pi} \]  

(31a)

\[\dot{\xi} = \sum_{i=1}^{m} \gamma^i \frac{\partial \Phi_i}{\partial q} \]  

(31b)

The computation of the plastic multipliers in (31) above is carried out according to the loading/-unloading and consistency conditions. The different results presented in the 2D context take the same form in the generalized stress and strain formulation.

### 3.1 Elastic domain

The elastic domain is here defined by four independent yield surfaces \( \Phi_i (\chi, \chi^p, \xi), 1 \leq i \leq 4 \), which are directly defined in generalized strain space \( \chi, \chi^p \) and \( \xi \). We denote as \( \epsilon^+ = \epsilon + \frac{1}{2} \kappa \) and \( \epsilon^- = \epsilon - \frac{1}{2} \kappa \) the local strain tensors at the top and bottom surfaces of plates and their spectral decompositions as: \( \epsilon^{i/+} = \sum_{k=1}^{2} \epsilon_k^{i/+} \otimes \pi_k^{i/+} \). By generalizing the presented 2D plasticity model to shells, we assume that the stress resultant plasticity starts as soon as the stress at either upper or lower face reaches the given elastic limit, leading to four different yield surfaces:

\[
\begin{align*}
\Phi_1 &= 2 \left( K + \frac{G}{3} \right) (\epsilon^1_+ - \epsilon_y) - k(\xi^1) \\
\Phi_2 &= 2 \left( K + \frac{G}{3} \right) (\epsilon^2_+ - \epsilon_y) - k(\xi^2) \\
\Phi_3 &= 2 \left( K + \frac{G}{3} \right) (\epsilon^3_+ - \epsilon_y) - k(\xi^3) \\
\Phi_4 &= 2 \left( K + \frac{G}{3} \right) (\epsilon^4_+ - \epsilon_y) - k(\xi^4)
\end{align*}
\]

(32a) - (32d)

With the use of state laws in (27) and the normalized values of stress resultants \( \hat{N} = N/h, \hat{M} = 6M/h^2, \hat{\pi} = \left( \hat{N} \hat{M} \right) \), we can write the four surfaces in stress resultants space \( \Phi_i (\hat{\pi}, \hat{q}) = \Phi_i (\chi, \chi^p, \xi) \):

\[
\begin{align*}
\Phi_1 &= \frac{K + 4G/3}{2G} \left| \hat{N} + \hat{M} \right|_1 - \frac{K - 2G/3}{2G} \left| \hat{N} - \hat{M} \right|_{11} - (\pi_y - q^+) \\
\Phi_2 &= -\frac{K - 2G/3}{2G} \left| \hat{N} + \hat{M} \right|_1 + \frac{K + 4G/3}{2G} \left| \hat{N} - \hat{M} \right|_{11} - (\pi_y - q^+) \\
\Phi_3 &= \frac{K + 4G/3}{2G} \left| \hat{N} - \hat{M} \right|_1 - \frac{K - 2G/3}{2G} \left| \hat{N} + \hat{M} \right|_{11} - (\pi_y - q^-) \\
\Phi_4 &= -\frac{K - 2G/3}{2G} \left| \hat{N} - \hat{M} \right|_1 + \frac{K + 4G/3}{2G} \left| \hat{N} + \hat{M} \right|_{11} - (\pi_y - q^-)
\end{align*}
\]

(33a) - (33e)

in which \( | \bullet |_{1/11} \) are the two principal values of the second-order symmetric tensor. The graphical illustration of this kind of criterion is given in Figure 6.

### 4 Discrete formulation

In this section, we recast this non-linear evolution problem in a discrete pseudo-time setting produced by a one-step scheme. The latter amounts to taking into account the values of evolution
variables at time \( t_n \), \( \chi^n \), \( \chi^p \), and \( \xi^n \), and computing the generalized strain increment \( \Delta \chi_{n+1} \) as well as applying an implicit backward-Euler integration scheme to obtain the corresponding values of internal variables.

For the best iterative value of strain increment, the computation of this kind starts by assuming the elastic trial state:

\[
\begin{align*}
\Delta \chi^{e,trial}_{n+1} &= \chi^{e}_{n} + \Delta \chi_{n+1} \\
\Delta \chi^{p,trial}_{n+1} &= \chi^{p}_{n} \\
\Delta \xi^{trial}_{n+1} &= \xi_{n}
\end{align*}
\]  

(34a, 34b, 34c)

which further leads to the corresponding values of stress resultants:

\[
\begin{align*}
\pi^{trial}_{n+1} &= \pi_{n} + \Xi: \Delta \chi_{n+1} \\
q^{trial}_{n+1} &= -k \left( \xi_{n}, \chi^{e,trial}_{n+1} \right)
\end{align*}
\]  

(35a, 35b)

If we finally find that all yield functions remain negative,

\[
\Phi^{trial}_{i,n+1} = \Phi_{i} \left( \pi^{trial}_{n+1}, q^{trial}_{n+1} \right) \leq 0 \quad 1 \leq i \leq 4
\]  

(36)

the trial stress state is admissible and it is accepted as final. The step is thus elastic. In the opposite, we have to compute the non-zero values of plastic multipliers in order to recover the admissibility of stress.

As discussed above, the key problem consists in determining the final set of active surfaces (with \( \lambda^{i} > 0 \)) according to the fact that \( \Phi^{trial}_{i,n+1} > 0 \Leftrightarrow \lambda^{i} > 0 \). In that sense, for the presented model, it should be emphasized that the second (resp. fourth) surface could be active even if \( \Phi^{trial}_{2/4,n+1} \leq 0 \).

Considering an active set of surfaces, the consistency condition leads to the non-linear system:

\[
\Phi_{i,n+1} \left( \pi, q \right) = 0
\]  

(37)

with the relations:

\[
\begin{align*}
\pi_{n+1} &= \pi^{trial}_{n+1} - \Xi \sum_{i=1}^{m} \lambda^{i} \frac{\partial \Phi_{i}}{\partial \pi}_{n+1} \\
q_{n+1} &= -k \left( \xi_{n+1}, \chi^{e,trial}_{n+1} \right)
\end{align*}
\]  

(38a, 38b)
with plastically admissible stress resultants and the plastic multipliers $\lambda^i$ as unknowns. Contrary to a single surface plasticity model for plates (see [IBR93]), the present case cannot be reduced to a single equation. We thus carry out an iterative solution for the complete system; the latter employs the following linearized form:

$$L [\Phi_{i,n+1}^{(k+1)} (\pi, q)] = \Phi_{i,n+1}^{(k)} (\pi, q) + \frac{\partial \Phi_i}{\partial \lambda_i}^{(k)} \Delta \lambda_{n+1}^{(k+1)} = 0 \quad (39)$$

$$\frac{\partial \Phi_i}{\partial \pi} = \frac{\partial \Phi_i}{\partial \pi} \cdot \frac{\partial \pi}{\partial \lambda_j} + \frac{\partial \Phi_i}{\partial q} \cdot \frac{\partial q}{\partial \lambda_j}$$

$$= - \frac{\partial \Phi_i}{\partial \pi} : \Xi \sum_{j=1}^m \frac{\partial \Phi_j}{\partial \pi} + \frac{\partial \Phi_i}{\partial q} \left( -D \cdot \frac{\partial \chi}{\partial \lambda_j} - U : \frac{\partial \chi}{\partial \lambda_j} \right)$$

$$= -g_{ij} \quad (40)$$

The matrix of the linearized system, with the components $g_{ij} = \left( \frac{\partial \Phi_i}{\partial \pi} \cdot \frac{\partial \Phi_j}{\partial \pi} - \frac{\partial \Phi_i}{\partial q} \cdot \frac{\partial \Phi_j}{\partial q} \right)$, remains positive-definite, which provides the guarantee for the solution.

Once the computation of the internal variables and plastically admissible stress resultants is completed, we turn to the solution of the global set of equilibrium equations. The key ingredient in that sense pertains to the elastoplastic consistent tangent modulus, which can be written as,

$$d\pi = \Xi : (d\chi - d\chi^p)$$

$$= \Xi^* : \left( d\chi - \sum_i d\lambda^i \frac{\partial \Phi_i}{\partial \pi} \right) \quad (41)$$

where $\Xi^* = \left[ \Xi^{-1} + \sum_i d\lambda_i \frac{\partial \Phi_i}{\partial \pi} \right]^{-1}$ is the algorithmic modulus. The latter is different from the continuum modulus (see [SIM85]) due to the rotation of principal directions.

The system of equations $d\Phi_i = 0$ provides the result:

$$d\gamma^i = \sum_j g^{ij} \frac{\partial \Phi_j}{\partial \pi} : \Xi^* : d\chi$$

which finally leads to,

$$d\pi = \Xi^{ep} : d\chi \cdot \Xi^{ep} = \Xi^* - \sum_{i,j} g^{ij} \left( \Xi^* \cdot \frac{\partial \Phi_i}{\partial \pi} \right) \otimes \left( \Xi^* \cdot \frac{\partial \Phi_j}{\partial \pi} \right) \quad (43)$$

### 5 Numerical examples

In this section we present several numerical examples, in order to illustrate the predictive capabilities of the presented model. All the computations are carried out by using the finite element program FEAP [ZIE00].
5.1 Illustrative example with simple test case

Considering pure tension and compression tests, the key point is to show the influence of the evolution of the fracture energy according to the stress state (see (10)). Figure 7 shows the uniaxial response of the proposed model for both cases; The first one corresponds to the "uncoupled" case for which fracture energy is assumed to remain constant for all possible stress states. Then, the second is a "coupled" case, where fracture energy is assumed to evolve according to expression (11).

![Figure 7: Uniaxial stress/strain response](image)

It is clearly shown in Figure 7 that the uncoupled case leads to a non-realistic behavior according to the fact that the amount of dissipated energy in both pure tension and compression is the same. Contrary to this situation, the coupled case allows to retrieve a physically-based uniaxial behavior by modifying the fracture energy. We reiterate on the point that this ingredient is crucial in order to obtain a representative behavior of brittle material.

5.2 Engineering application - chimney under fire

In this example we present an industrial application dealing with the fire resistance of a chimney. The latter is made of an assembly of hollow clay blocks (see Figure 8). The inner part of these blocks is submitted to the flow of hot gas (up to 1000 °C in 10 mn). This heating drives to the failure of the blocks and the possibility for the gas to flow outside from the chimney. According to the industrial point of view, the key point is to evaluate the time before failure.

![Figure 8: Assembly of hollow clay blocks](image)

In order to compare the numerical results with the corresponding experimental data, a test has been driven on a chimney made of three hollow blocks. Thermocouples and strain gauges have been put on the block placed in middle position. The former are providing temperatures through the thickness of the block and so furnish some information concerning temperature gradient. Strain gauges have been placed on the outer face only in order to estimate stresses evolution with respect to time.

By taking into account the symmetry, the numerical model for this example is made of 32 flat shell elements only. Comparing to classical solid three dimensional modelling, this application is thus clearly showing the main advantage of using shell elements by drastically reducing the size.
of the discrete problem. According to the loading case, a coupled thermomechanical analysis has been driven using these flat shell elements incorporating the mechanical model presented here. Concerning the heat transfer problem, a special discrete form of the transfer equation has been developed, allowing to take into account for both average temperatures and through-the-thickness gradients (see [COL04a]). This model is representing both conduction and radiative heat transfers which are leading to a nonlinear problem. The latter takes place inside each cell and it can be taken into account by classical eight nodes solid elements (see [COL04a]).

The boundary condition are chosen as a convective heat transfer between the hot gas and the inner face of the block. We chose the convection coefficient in order to reproduce the temperature evolution on this face observed during the test (see Figure 10).

The parameters used to drive the numerical analysis are shown in Table 1. For this example, we assumed that all these parameters remain constant with respect to temperature evolution.

Figure 10 shows the comparison between numerical analysis and experimental data. The first part deals with temperatures evolutions in time. The four curves are representing the surface temperature from the inner face to the outer face of the block. As already mentioned, the convection coefficient has been chosen in order to reproduce the inner face temperature evolution. According to this point, it is shown that the outer face temperature is not accurately predicted. The main reason for this is the choice of one shell element only in the heat flow direction. Nevertheless, the main objective of this computation is not to provide a detailed prediction of the temperature field (a three dimensional analysis with solid element would be more adequate) and the accuracy
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>masse density</td>
<td>1.870</td>
</tr>
<tr>
<td>heat capacity</td>
<td>836 J/kg °K⁻¹</td>
</tr>
<tr>
<td>thermal conductivity</td>
<td>0.45 W/m °K⁻¹</td>
</tr>
<tr>
<td>thermal expansion coefficient</td>
<td>7.10⁻⁶ °K⁻¹</td>
</tr>
<tr>
<td>Young’s modulus</td>
<td>12 GPa</td>
</tr>
<tr>
<td>Poisson ratio</td>
<td>0.2</td>
</tr>
<tr>
<td>( \sigma_y )</td>
<td>11 MPa</td>
</tr>
<tr>
<td>( G_{\text{I}} )</td>
<td>80 J/m²</td>
</tr>
<tr>
<td>( G_{\text{c}} )</td>
<td>10 Gt</td>
</tr>
</tbody>
</table>

Table 1: Clay block - parameters

obtained is sufficient for the mechanical analysis.

Concerning the mechanical behavior, the numerical analysis leads to failure which starts in the inner part of the corner of the block, and propagates to the outer part. This result is completely in accordance with the experimental behavior for which the crack pattern is almost vertical in two opposite corners. Figure 11 shows the main cracks in this region after the test.

Figure 10 also shows the horizontal stress evolution on the outer face of the block with respect to time. In order to compare with the experimental data provided by the strain gauge, the computed stress corresponds to the same point. It is clearly shown that the proposed model is able to reproduce with a good accuracy the mechanical behavior of this structure. The only discrepancy occurs in the first part of the heating process, where the numerical approach is overestimating the stress. This is mainly due to the overestimation of the temperature at the outer face (see Figure 10a). The time to reach the failure is well reproduced, around ten minutes for both cases. This finding is of crucial concern for the industrial application.

5.3 Thermomechanical analysis of hollow brick wall

In this example we consider a thermomechanical coupling in the cellular units placed with a brick wall. We assume that the geometry and the loading allow for exploiting the periodicity conditions. This implies that the analysis can be carried out on a single cellular unit isolated from the whole structure at the level of interface with neighboring units, by applying the corresponding
boundary conditions which assure periodicity. More precisely, for the typical unit assembly in a
brick wall (see Figure 12) with only partial overlapping of successive layers, the same periodicity
conditions are enforced only over half of the brick. Therefore, the domain which is retained in

![Clay block - cracks pattern and failure mechanism](image1)

![Unit assembly in a brick wall and periodic BC](image2)

The analysis corresponds to one typical unit of the size $570 \times 200 \times 200 \, mm^3$. This domain also
includes a half of the vertical and horizontal joints with the thickness equal to $10 \, mm$.

The chosen finite element mesh (see Figure 13) consists of three vertical vertical layers of flat
shell element which brings the total number of these element to 384 for the entire brick. Another
subtlety of the model is the choice which is made for the representing the interface joints. This one
is model by elastic solid elements covering the cells of the brick placed only at the top. However,
the latter does not introduce any non-symmetry in the problem, considering the periodicity in the
boundary conditions.

Both mechanical and thermal loading is applied in this case. The mechanical loading is supposed
to represent the dead load on the brick chosen as a compressive loading of $1.3 \, MPa$, which is
introduced directly at the level of each element as the initial compressive loading in the bricks,
remaining constant afterwards. The thermal loading is then applied, in terms of the uniform
temperature field applied only at the brick facet exposed to fire. The time evolution of this
temperature field is given as

$$\theta(t) = \theta_0 + 345 \log(8t + 1) \quad (44)$$

where $\theta_0$ is the initial temperature and $t$ the time in minutes.

The mechanical and thermal properties of the brick material are chosen as given in Table 2. The properties of the interface are given in Table 3.

### Table 2: Mechanical and thermal properties of the brick

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>density</td>
<td>1.870</td>
</tr>
<tr>
<td>heat capacity</td>
<td>836 $J.kg^{-1}.K^{-1}$</td>
</tr>
<tr>
<td>conductivity (parallel to flakes)</td>
<td>0.55 $W.m^{-1}.K^{-1}$</td>
</tr>
<tr>
<td>conductivity (perpendicular to flakes)</td>
<td>0.35 $W.m^{-1}.K^{-1}$</td>
</tr>
<tr>
<td>thermal expansion coefficient at $\theta_{ref}$</td>
<td>7.10$^{-6}$ $K^{-1}$</td>
</tr>
<tr>
<td>Young modulus</td>
<td>12 $GPa$</td>
</tr>
<tr>
<td>Poisson ratio</td>
<td>0.2</td>
</tr>
<tr>
<td>$\sigma_y$ at $\theta_{ref}$</td>
<td>14.5 $MPa$</td>
</tr>
<tr>
<td>fracture energy</td>
<td>80 $J.m^{-2}$</td>
</tr>
</tbody>
</table>

### Table 3: Mechanical and thermal properties of the interface

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>density</td>
<td>2.100</td>
</tr>
<tr>
<td>heat capacity</td>
<td>950 $J.kg^{-1}.K^{-1}$</td>
</tr>
<tr>
<td>conductivity</td>
<td>1.15 $W.m^{-1}.K^{-1}$</td>
</tr>
<tr>
<td>thermal expansion coefficient</td>
<td>1.10$^{-5}$ $K^{-1}$</td>
</tr>
<tr>
<td>Young modulus</td>
<td>15 $GPa$</td>
</tr>
<tr>
<td>Poisson ratio</td>
<td>0.25</td>
</tr>
</tbody>
</table>

![Figure 13: FE mesh for a cellular unit](image)

The mechanical and thermal properties of the brick material are chosen as given in Table 2. The properties of the interface are given in Table 3.
First, the results are presented in terms of temperature field. Figure 15a shows the evolution of the temperature in three different cells (see Figure 14 for locations). The experimental results are provided by thermocouples inside the cells. Therefore, we compare these values with the temperatures of the two surfaces on both sides of each cell obtained by the finite element analysis. The comparison shows we are able to capture the temperature evolutions even far from the exposed face of the wall. This result is confirmed by Figure 15b which shows a temperature profile 48 minutes after the beginning of heating. The key point in order to obtain such good result is the introduction of radiative exchanges in the heat transfer model.

On the mechanical point of view, Figure 16a shows the evolution of the sum of vertical reactions at selected nodes. Each curve corresponds to a line of nodes parallel to the exposed face of the wall and positive values are for compression (with prestressed initial value due to mechanical constant loading). Figure 16b shows the comparison on the horizontal displacement of wall built with ten rows of bricks. This bending is due to the temperature gradient through the wall. We show that the stiffness provided by the analysis is quite correct even if the displacement are slightly overestimated by the absence of mechanical boundary conditions. Global analysis on entire wall (e.g. without periodic boundary conditions) have been made in order to improve this result.

![Figure 15: Brick – temperature evolutions and profile after 48 mn](image1)

![Figure 16: Brick – total vertical reactions for the first lines and horizontal displacement](image2)
6 Conclusions

The multi-surface model proposed herein finds its place quite naturally within the multiscale approach for modelling the cellular structures (e.g. see [IBR05]). The model is used for the computations at the finest scale to explain the physical nature of damage mechanisms. For that reason with a large number of computations to be performed at the finest scale, the numerical implementation of the model is carried out with a great care in order to ensure the most robust model performance.

The original feature of the model concerns the plasticity criterion proposed in strain space, with respect to the principal values of elastic strain tensor. The latter allows to represent the cracking phenomena both in tension and in lateral straining under compression. We thus provide quite a realistic description of cracking phenomena in brittle materials, with a very few parameters. As such, the model can be applied to other situation where the brittle rupture is a dominant failure mode.

Acknowledgements:
This work was supported by the French Ministry of Research and CTTB. This support is gratefully acknowledged.

References


[CRIS95] Crisfield M. Stress resultant plasticity for shells, Proceedings of COMPLAS 5, Barcelona, 1995


18
