Microscale and mesoscale discrete models for dynamic fracture of structures built of brittle material

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Abstract

In this work we present the discrete models for dynamic fracture of structures built of brittle materials. The models construction is based on Voronoi cell representation of the heterogeneous structure, with the beam lattice network used to model the cohesive and compressive forces between the neighboring cells. Each lattice component is a geometrically exact shear deformable beam which can describe large rigid body motion and the most salient fracture mechanisms. The latter can be represented through the corresponding form of the beam constitutive equations, which are derived either at microscale with random distribution of material properties or at a mesoscale with average deterministic values. The proposed models are also placed within the framework of dynamics, where special attention is paid to constructing the lattice network mass matrix as well as the corresponding time-stepping schemes. Numerical simulations of compression and bending tests is given to illustrate the models performance.

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1. Introduction

The characteristic inelastic response of heterogeneous brittle materials, such as concrete, ceramics or metallic powder, is very difficult to interpret without appealing to their micro-structure. The standard test procedures to obtain the mechanical properties of such materials, for example simple traction or compression test or yet a 3-point bending test, show a significant dispersion of experimental results, regarding not only local properties such as crack spacing and characteristic profile but also global response represented with the resulting force-displacement diagram. In order to develop a clear interpretation of these experimental results and predictive models for the heterogeneous brittle material behavior the usual framework of continuum mechanics ought to be abandoned in favor of so-called discrete models which start from the smallest computationally acceptable scale of micromechanics, where a heterogeneous material is considered as an assembly of elementary particles (in general represented by rigid bodies) held together by cohesive forces. This kind of approach, pioneered by Cundall in late 70s (e.g. see [9]), has until recently been mainly restricted to granular soils (e.g. see [6] for a recent review), where material is considered as assembly of (spherical) particles whose interaction is represented by frictional contact models of Mohr–Coulomb type. More recently, these models are extended to other heterogeneous brittle materials, introducing a number of modifications concerning the shape of interaction particles, the presence of cohesive forces and the possible mechanisms representing the fracture. Some of the representative works of this kind are those of Kun and Herrmann [18], d’Addetta et al. [1], Schlangen and Garboczi [25] or Chang et al. [8] who all use the Euler–Bernoulli beam lattice network to represent cohesive forces, the works of Camacho and Ortiz [7] or Ortiz and Pandolfi [21] who developed contact elements for modeling of cohesive forces in 2D and 3D fracture problems.

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or yet the work of Belytschko and Organ [4] who include the cohesive forces within the framework of element-free Galerkin method.

With respect to these works the main novelties introduced herein concern the following.

(i) The construction of the beam lattice network representing the cohesive forces which employs the geometrically nonlinear beam model of Reissner (e.g. [24]); the beam model of this kind, which includes the shear deformation, allows to easily represent all the fracture mechanisms leading to either crack opening displacement or crack sliding displacement, the latter not being available for Euler–Bernoulli beam lattice. Moreover, the ability of the present model to handle large rotations is important for representing the behavior of a heavily damaged zone as well as the displacement patterns of an assembly of several connected particles which split from the main structure. The constitutive model for cohesive forces obtained from analysis of interface between the neighboring particles can be implemented within the framework of reduced integration of Reissner’s beam (e.g. see [15]).

(ii) 1D discrete interface models of Delaplace et al. [10] where the mechanical properties are considered as random variables, are here extended to structures with an arbitrary crack orientation; it is shown that the analysis of this kind can be brought to bear upon the identification of intrinsic material properties, such as internal length parameter for fracture, and development of irreversible damage phenomena; furthermore, the results obtained by this kind of models pertaining to instability producing localisation problems can be interpreted through the appropriate studies of response fluctuation and statistical distribution of avalanches (e.g. see [14,22] or [11]).

(iii) The constantly increasing demand to generalize the heterogeneous models of this kind in order to make them applicable in the analysis of structures requires establishing the link between the micromechanics models of this kind with the standard phenomenological models developed at macroscale. This work can be carried out without major difficulties (e.g. see [8]) only for elastic response; the fracture process can no longer be represented in detail at macroscale, where crack patterns are ‘smeared’ and artificial characteristic length parameters must be introduced. The potential direction shortly explored in this work concerns the models of beam lattice network constructed at mesoscale, where both elastic and inelastic fracture response can be developed in a systematic manner.

The outline of the paper is as follows. In the next section, we present the governing equations of the geometrically nonlinear beam model of Reissner used as a component of the beam lattice network. The constitutive models for fracture are discussed in Section 3, both at microscale and mesoscale. The dynamic fracture of the beam lattice network and the time-stepping schemes are discussed in Section 4. In Section 5, we present the results of illustrative numerical computations. The concluding remarks are given in Section 6.

2. Geometrically exact shear deformable beam as a component of lattice network

In order to provide a reliable description of a complex heterogeneous structure, we make use of Voronoi cells, with each cell represented as a convex polygon. The Voronoi polygon construction can either be obtained from a random tessellation of the plane domain occupied by the structure or as a more or less reliable scanner-like representation of the observed heterogeneities of a structural assembly to be tested. In either case, one starts by choosing a set of points in the plane and then assigning to each point the part of the plane domain which is closer to it than to any other of the chosen points. For the given set of points \( P_i, i = 1, \ldots, n \) the plane domain \( \Omega \) will be covered by non-overlapping polygons \( \Omega = \cup \Omega_i \), where each polygon \( \Omega_i \) associated with point \( P_i \) is defined as

\[
\Omega_i = \{ P \text{ such that } d(P, P_i) \leq d(P, P_j) \quad \forall j \neq i \} \tag{1}
\]

where \( d(\cdot, \cdot) \) denotes the usual distance in the Euclidean space (see Fig. 1). It follows from (1) that each side of a

Fig. 1. (a) Voronoi polygon representation of domain \( \Omega \) and (b) two neighboring Voronoi cells.
The given polygon splits the distance \(d(P_i, P_j) = l_{ij} = l^r\) in half, which is an important observation to be exploited later.

The lattice network representing the heterogeneous structure is constructed by connecting all the points in neighboring Voronoi polygons by 2-node straight beam elements, which leads to Delaunay triangulationization (e.g. see [13]) where the beam elements are placed along the sides of each triangle. In the rest of this section, we briefly describe the chosen geometrically nonlinear model of shear deformable beam. For a more elaborate description we refer to [2,16,24].

In the local coordinate system (see Fig. 1) we describe the initial and deformed configurations of the beam as

\[(x_1 + y_2) \rightarrow \phi = (x + u)e_1 + ve_2 + yt_2\]  

(2)

where

\[t_2 = Ae_2, \quad \Lambda = \begin{bmatrix} \cos \psi & -\sin \psi \\ \sin \psi & \cos \psi \end{bmatrix}\]

\(u\) and \(v\) are displacement components of the beam axis and \(t_2\) is the unit vector indicating the position of the beam section (see Fig. 2). According to the kinematic hypothesis of Reissner [24], the plane section remains plane, but not necessary perpendicular to the beam axis if one also wants to take the shear into account. The corresponding strain measure for dilatation and shear indicated in Fig. 2 can then be written as

\[\left( \begin{array}{c} e' \\
\gamma \end{array} \right) = \left( \begin{array}{c} 1 + u' - \cos \psi \\
v' - \sin \psi \end{array} \right) \iff \varepsilon = \varphi - \mathbf{t}_1\]  

(3)

where \((\cdot)' = \partial(\cdot)/\partial x\).

We note in passing that if the beam displacements and rotations remain small, such that \(\cos \psi \approx 1\) and \(\sin \psi \approx \psi\), we obtain from (3) above the standard format of the strain measures for the Timoshenko beam (see e.g. [15]). For large displacements and rotations, there exist many alternative forms for expressing the strain measures (e.g. see [2,20]) with the Biot strains as the most suitable for this kind of beam theory (e.g. see [15])

\[H = \mathbf{A}^T \mathbf{F} - \mathbf{I}_2, \quad \mathbf{F} = \begin{bmatrix} \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{bmatrix} = [\phi'; t_2]\]  

(4)

where

\[\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\
0 & 1 \end{bmatrix}\]

The Biot strain tensor in (4) above can also be expressed in terms of components

\[H_{11}(x,y) = \Sigma(x) - yK(x)\]

\[H_{21}(x,y) = \Gamma(x)\]

(5)

where

\[\Sigma(x) = t_1 \cdot \phi' - e_1 \cdot e_1, \quad K(x) = t_2 \cdot t_1,\]

\[\Gamma(x) = t_2 \cdot \phi' - e_2 \cdot e_1\]

The Biot stress tensor components (e.g. [5,15]) are employed as work-conjugate to the Biot strain in (5) above. For example, for a linear elastic behavior of the beam we have

\[T_{11}(x,y) = E(x,y)H_{11}(x,y),\]

\[T_{21}(x,y) = G(x,y)H_{21}(x,y)\]  

(6)

where \(E\) and \(G\) are, respectively, Young’s and shear moduli.

In following the standard developments for beam theories, we can further define the stress resultants \(N\) and \(V\) and couple \(M\) according to

\[N = \int_{A'} T_{11} \, dA, \quad V = \int_{A'} T_{21} \, dA,\]

\[M = -\int_{A'} yT_{11} \, dA\]  

(7)

where \(A'\) is the area of the beam cross-section. If the cross-section is symmetric and the local reference frame is placed in the center of gravity, we can rewrite the elastic constitutive equations in (6) directly in terms of the stress resultants with

\[N = EA'\Sigma, \quad V = GA'\Gamma, \quad M = EF'K\]  

(8)

where \(I' = \int_{A'} y^2 \, dA\) is the section moment of inertia.
With this kind of work-conjugate pairs we can write the total potential energy of the elastic beam as

\[ P_e = \frac{1}{2} \int \left[ \Sigma (CA' \Sigma + \Gamma (G A' \Gamma) + K (E F') K) \right] dx - P_{e_{\text{ext}}} \quad (9) \]

With the main goal in mind of developing the discrete model of heterogeneous structure employing the beam lattice network, we can use the finite element discretisation of the presented beam model to provide the corresponding representation of lattice network component. In particular, by assuming the simplest choice of finite element approximations, we assume that each strain component in (5) remains constant along the beam, which allows us (e.g. see [15]) to express them as functions of the beam end displacements and rotations according to

\[
\begin{align*}
\Sigma &= \cos \left( \frac{\psi_1 + \psi_2}{2} \right) \left( 1 + \frac{u_2 - u_1}{I} \right) + \sin \left( \frac{\psi_1 + \psi_2}{2} \right) \left( \frac{v_2 - v_1}{l} \right) - 1 \\
\Gamma &= -\sin \left( \frac{\psi_1 + \psi_2}{2} \right) \left( 1 + \frac{u_2 - u_1}{I} \right) + \cos \left( \frac{\psi_1 + \psi_2}{2} \right) \left( \frac{v_2 - v_1}{l} \right) \\
K &= \frac{\psi_2 - \psi_1}{2}
\end{align*}
\]

One can thus easily show that the beam length change can be expressed with

\[ \left( \frac{\Delta s}{l} \right) = (1 + \Sigma)^2 + \Gamma^2 \quad (11) \]

which is illustrated in Fig. 3. We note that the strain measures in (10) can also be obtained by using the reduced, one-point Gauss integration rule (see [15]) along with the standard isoparametric interpolations. We also note that the reduced Gauss integration rule makes use of the cross-section placed at the mid-span of the beam, which can be exploited to accommodate within the same framework the developments of the beam element arrays for more elaborate constitutive models allowing for inelastic behavior as shown in the next section.

3. Constitutive models for fracture through lattice network

In this section we present the constitutive model of brittle fracture implemented within the framework of the beam lattice network. The model is first developed at the microscale, where each beam is considered to be representation of only two elementary fracture modes, measured by either beam length change or by relative rotations. The mechanical properties of the beam components of the lattice network are considered to be random variables with the Gaussian density distribution. The alternative form of the constitutive model is also developed at the mesoscale where the mechanical properties are average values of those from the microscale and are therefore considered as deterministic (the constitutive model of this kind can be obtained by integrating through the thickness).

3.1. Microscale constitutive modeling

The main hypothesis in constructing the microscale constitutive model is that the size of the each Voronoi cell corresponds to the representative size of heterogeneities, or otherwise, each cell is of the characteristic of a single grain. Like grains, the Voronoi cells are kept together by cohesive forces. These cohesive forces are represented by the beam lattice network, which are constructed so that the beams connect the center of gravity of each pair of neighboring cells. In other word, contrary to the arrangement shown in Fig. 1b, \( P_i \) and \( P_j \) are placed in the center of gravity of their cells (see Fig. 4). This kind of choice allows us to consider that the elastic response of each beam is given by a diagonal matrix, as already indicated in (8), with the corresponding values of area and moment of inertia for the beam cross-section computed from the length \( h_{ij} = h' \) of the common size of the neighboring cells taken as the fictitious thickness, to obtain

![Fig. 3. Graphic interpretation of the Biot strain measures.](image-url)
The elastic limits restricted to and second, the relative bending deformation is restricted to

\[ \frac{(1 + \Sigma)^2 + \Gamma^2}{2} = \frac{\Delta \epsilon}{\epsilon_e'_{ij}} \leq \epsilon_{ij}' \quad (13) \]

and second, the relative bending deformation is restricted to

\[ |\mathbf{K}| = \left| \frac{\psi_j - \psi_i}{\epsilon_e'} \right| \leq \theta_{ij}' \quad (14) \]

The chosen elastic constitutive response of each beam representing cohesive forces is limited to elastic domain. Outside that domain the fracture is assumed, with all the cohesive forces and moments being reduced instantaneously to zero. The fracture criterion is set in the strain space. It takes into account two possible modes of fracture: first the traction induced separation between the neighboring cells is limited to representing cohesive forces is limited to elastic domain. The mean value of the fracture deformation \( \epsilon_{ij}' \) controls the global constitutive behavior in traction where the rupture of all the cohesive links or cracks are assumed to come in contact in the later stage of the deformation thus not connected by cohesive beam components, but can further buckle and break. The chosen rotation limit will thus control post-peak softening constitutive behavior.

The final ingredient of the constitutive model at microscale takes into account the interaction of the Voronoi cells which are not initially the neighbors and thus not connected by cohesive beam components, but can come in contact in the later stage of the deformation with extensive spreading of the fracture zone. The penalty like frictionless contact model is used for such a purpose, where the cell interpenetration is allowed with the contact force proportional to the overlapping area applied in the direction connecting the centers of gravity for two cells (see Fig. 5).

The mean value of the relative rotation \( \theta_{ij}' \) is related to the case of compressive loading where cracks might appear parallel to the loading direction due to the Poisson’s ratio effect, where so-created isolated beam components can further buckle and break. The chosen rotation limit will thus control post-peak softening constitutive behavior.

The inelastic response of the beam is also computed using the one-point Gauss quadrature used to compute the beam element stiffness matrix and residual vector, we ought to assure that the considered cross-section is placed in the middle of the beam span, which implies that original points \( P_i \) and \( P_j \) are kept as the nodes of the beam lattice network. This also implies that the beam reference frame no longer passes through the center of gravity of the section. The elastic response of the beam will thus introduce the coupling of axial and bending strains as

\[
\begin{bmatrix}
N \\
V \\
M
\end{bmatrix} = \begin{bmatrix}
EA' & 0 & ES' \\
0 & GA' & 0 \\
ES' & 0 & EI'
\end{bmatrix} \begin{bmatrix}
\Sigma \\
\Gamma \\
K
\end{bmatrix}
\]

where, for the section of unit width, we can write

\[
A' = \int_{-h}^{h} y \, dy = h', \quad S' = \int_{-h}^{h} y^2 \, dy = \frac{1}{2} [(h')^2 - (h' - h')^2],
\]

\[
I' = \int_{-h}^{h} y^2 \, dy = \frac{1}{12} [(h')^3 - (h' - h')^3]
\]

The chosen inelastic constitutive response (and the tangent modulus) be obtained for all the values of the natural coordinate \( \eta \) in through-the-thickness direction in accordance with the chosen numerical integration rule. In particular, for
any chosen value \( \eta_l, l = 1, \ldots, n_{int} \), we can compute the relative normal and tangential displacements by making use of the results in (5) and (10) as

\[
\delta_x(\eta_l) = (\Sigma + y(\eta_l)K)t^e \\
\delta_t(\eta_l) = \Gamma t^e
\]  

(17)

We can then follow the idea of Jeng and Shah [17], which is also used in [4,7] or [21], to combine the crack opening displacement \( \delta_x(\eta_l) \) and crack sliding displacement \( \delta_t(\eta_l) \) into effective opening displacement

\[
\delta(\eta_l) = \sqrt{\beta_2^2[\delta_x(\eta_l)]^2 + [\delta_t(\eta_l)]^2}
\]  

(18)

where parameter \( \beta \) can be varied with respect to the default unit value in order to assign different weights to each component of crack displacement. The mixed mode cohesive law can thus be constructed to relate the effective displacement \( \delta(\eta_l) \) and the effective traction

\[
t(\eta_l) = \sqrt{\beta_2^2[t(\eta_l)]^2 + [s(\eta_l)]^2}
\]  

(19)

In constructing this kind of traction–displacement law we ought to take into account its compatibility with the elastic response in (15) and the irreversible nature of fracture process. The former is achieved by setting the fracture traction value \( t_0 \) and the latter is described through the loading/unloading conditions

\[
\delta' = \frac{\delta \phi}{\partial t}, \quad \gamma \geq 0, \quad \phi \leq 0, \quad \gamma \phi = 0
\]  

(20)

where \( \phi(t, \gamma) \) is the fracture criterion given as

\[
\phi(t, \gamma) = t - [t_j - g(\xi)]
\]  

(21)

In (21) above, we denote by \( g \) the stress-like variable conjugate to the internal variable \( \xi \) which describes the softening behavior (see Fig. 6). The traction–displacement constitutive model can be written in the rate-form within the standard format of damage models according to

\[
t(\eta_l) = \begin{cases} 
    D^{-1}\dot{\delta}; & \phi < 0 \\
    \frac{D^{-1}1}{\frac{D^{-1}1}{K(\xi)}}\dot{\delta}; & \phi = 0, \quad \dot{\phi} = 0
\end{cases}
\]  

(22)

where \( D \) is the current value of the compliance, i.e. with \( \delta = Dt \). Similar computations as those described in (17)–(22) can be carried out for the remaining numerical integration points resulting with the corresponding element contributions to tangent stiffness and residual force vector. The stress resultant and couples are thus obtained by integrating different contributions numerically (instead of analytic result in (7)) by making use of the computed values \( t(\eta_l) \) according to the given quadrature rule.

4. Dynamics of lattice network and time-stepping schemes

For any of the fracture models developed in the previous section, we can formally write the equation expressing the equilibrium of the heterogeneous structure under quasi-static loading according to

\[
0 = \delta\Pi(\Sigma, \Gamma, K)(u^*, v^*, \psi^*)
\]

\[
= \int \left( \Sigma^* N + \Gamma^* V + K^* M \right) dx
\]

\[
- \delta\Pi_{ext}(u^*, v^*, \psi^*)
\]  

(23)

where \( u^*, v^* \) and \( \psi^* \) are virtual displacements and rotations, whereas the \( \Sigma^*, \Gamma^* \) and \( K^* \) are virtual strain measures. The latter can be obtained as the Gateaux derivative of the real strain measures according to

\[
K^* = \left. \frac{d}{dx} \right|_{x=0} \left[ \psi' + \chi\psi^* \right] = \psi'^* \\
\left( \Sigma^* \right)^T = \left. \frac{d}{dx} \right|_{x=0} \left[ \Lambda^T(\psi')\phi' - e_i \right] = \Lambda^T(u^*\phi' + W^T\phi'\psi^*)
\]  

(24)

where

\[
W = \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
\]

Introducing these results into (23) and using one-point Gauss quadrature along the beam length we can obtain the internal forces contribution of beam element \( e \) to the discrete form of the equilibrium equation according to

\[
\left[ u_i^* T v_i^* \right]_{int} = \left( \begin{array}{c}
u_i^* \\
\psi_i^*
\end{array} \right) = \begin{bmatrix}
(-1)^i I_2 & 0 \\
0 & I^e/2
\end{bmatrix} \left[ \begin{array}{c}
I^e_2 \\
(\phi')^T W
\end{array} \right] \\
\times (Nt_i + Yt_2) + \left( \begin{array}{c}
0 \\
0
\end{array} \right) M
\]  

(25)

Before processing the contributions from all beam elements through the finite element assembly procedure (e.g. see [27] or [3]) the internal force vector in (25) is
expressed in the global coordinate system adopted for the whole structure (see Fig. 1)

\[
\mathbf{u}_{\text{ext}} = \begin{bmatrix} A^1_i \\ 0^1 \\ 1 \end{bmatrix} (\mathbf{U}^i_{\text{ext}}) \Rightarrow \mathbf{F}_{\text{P}_j}^{\text{int}} = \begin{bmatrix} A^1_i \\ 0^1 \\ 1 \end{bmatrix} \mathbf{F}_{\text{P}_j}^{\text{ext}} \quad (26)
\]

where

\[
\begin{align*}
\mathbf{A}_0 &= \begin{bmatrix} \cos \varphi' - \sin \varphi' \\ \sin \varphi' \cos \varphi' \end{bmatrix} ; \\
\sin \varphi &= \frac{Y_{\text{P}_j} - Y_{\text{P}_i}}{l_{ij}}
\end{align*}
\]

The second term on the right-hand side in (28) above gives rise to the tangent stiffness matrix. By transforming those components into the global coordinate system as shown in (26) we can obtain the linearized equilibrium equations

\[
\sum_{e=1}^{n_{\text{el}}} \left\{ \sum_{j} \mathbf{K}_{\text{P}_j}^{e} \Delta \mathbf{U}_{\text{P}_j} = \mathbf{F}_{\text{P}_j}^{\text{ext}} - \mathbf{F}_{\text{P}_j}^{\text{int}} \right\} ; \quad \forall P_i = \mathbf{c} \mathbf{m}^e(i)
\]

(30)

As described in the rest of this section, it is fairly easy to extend the proposed framework to dynamics. Namely, it is precisely the main advantage of the chosen beam model for lattice network which concerns the use of the fixed inertia frame (e.g. see [26] or [16]) to develop the corresponding equations of motion for the dynamic case in the standard format; The latter implies the linear inertia term and constant mass matrix, with all nonlinearities hidden in the internal force vector, according to

\[
\sum_{e=1}^{n_{\text{el}}} \left\{ \sum_{j} \mathbf{M}_{\text{P}_j}^{e} \dot{\mathbf{U}}_{\text{P}_j} + \mathbf{F}_{\text{P}_j}^{\text{int}} = \mathbf{F}_{\text{P}_j}^{\text{ext}} \right\} ; \quad \forall P_i = \mathbf{c} \mathbf{m}^e(i)
\]

(31)

The mass matrix contribution of each beam element is computed in the diagonal form, where the mass of the corresponding triangular part of the Voronoi polygon is lumped at the node (see Fig. 7) to obtain

\[
\mathbf{M}_{\text{P}_j}^{e} = \text{diag}(m_{\text{P}_j}, m_{\text{P}_j}, J_{\text{P}_j}, m_{\text{P}_j}, m_{\text{P}_j}, J_{\text{P}_j})
\]

(32)

with

\[
m_{\text{P}_j} = \rho A_{12}\ell_j; \quad m_{\text{P}_j} = \rho A_{12}\ell_j
\]

where \( J_{\text{P}_j} \) and \( J_{\text{P}_j} \) are the chosen values of the momenta of inertia. The differential equations of motion in (31) are solved by using the Newmark time-integration scheme (e.g. see [3]). With this kind of scheme the central problem of computational dynamics can be presented as follows:

- Given: \( \mathbf{d}_n = (\mathbf{U}_n(t_n)), \quad \mathbf{v}_n = (\mathbf{U}_n(t_n)), \quad \mathbf{a}_n = (\mathbf{U}_n(t_n)) \) which satisfy the equations of motion and \( \Delta t = t_{n+1} - t_n \)
- Find: \( \mathbf{d}_{n+1}, \quad \mathbf{v}_{n+1}, \quad \mathbf{a}_{n+1} \) such that

\[
\mathbf{M}_{\text{d}_{n+1}} + \mathbf{F}_{\text{d}_{n+1}}^{\text{int}} = \mathbf{F}_{\text{d}_{n+1}}^{\text{ext}}
\]

Two possible implementations of the Newmark scheme can be considered.

Fig. 7. Assembly for mass matrix computations.
(i) trapezoidal rule (implicit scheme with $\beta = 1/4$ and $\gamma = 1/2$)

$$a_{n+1} = \frac{1}{\beta \Delta t} (d_{n+1} - d_n) - \frac{1}{\beta M} v_n + \frac{1 - 2\beta}{2\gamma} a_n$$

$$v_{n+1} = \frac{\gamma}{\beta M} (d_{n+1} - d_n) + \left(1 - \frac{2}{\beta}\right) v_n + \Delta t \frac{1 - \gamma}{2\beta} a_n$$

$$M a_{n+1} + F_{\text{int}} (d_{n+1}) = F_{\text{ext}}$$

(ii) central difference scheme (explicit scheme with $\beta = 0$ and $\gamma = 1/2$)

$$d_{n+1} = d_n + \Delta t v_n + \frac{\Delta t^2}{2} [(1 - 2\beta) a_n + 2\beta a_{n+1}]$$

$$v_{n+1} = v_n + \Delta t [(1 - \gamma) a_n + \gamma a_{n+1}]$$

$$M a_{n+1} = F_{\text{ext}} (d_{n+1})$$

The former scheme must further be solved by an iterative procedure along the same lines as the corresponding solution in (30) for the quasi-static case but with effective stiffness matrix and residual both accounting for the inertia term contributions. The latter scheme in (34) is especially advantageous for the chosen diagonal form of the mass matrix where the computation of the unknown acceleration vector is very simple to perform. Both schemes are implemented and tested in the numerical simulations described in the next section.

5. Numerical simulations

In this section we present the numerical results obtained for two different test problems. In both problems the selected structure is representative of basic experimental setup which is typically selected to evaluate the material properties of homogeneous materials, such as a simple tension test or three-point beam bending test. Both numerical simulations are performed with micro-scale model only, although for the chosen specimen dimensions and number of cells such a terminology might be justified mainly for the grain size typical of concrete structures. All the computations are performed by in-house computer program developed for simulation of discrete model response.

5.1. Simple tension test

In the first example, we choose to model a simple tension test, which is typically developed in order to identify material parameters such as the limit of elasticity or the fracture strength. The tested sample is a square of 10 cm long. The applied traction force in a simple tension test is replaced by an imposed displacement applied on the upper side of the specimen, thus simulating the displacement-controlled test where material behavior remains stable. The boundary conditions of fixed displacement are imposed on the bottom side. The material parameters are selected as

$$E = 30 \text{ GPa}, \quad G = 15 \text{ GPa}, \quad \rho = 23 \text{ kN/m}^3$$

Each breaking threshold in traction is a random variable obtained from a Gaussian distribution with mean value $\bar{\varepsilon} = 1.5 \times 10^{-4}$ and a standard deviation of 0.2. Breaking threshold in rotation is a random variable obtained from a Gaussian distribution with mean value $\bar{\theta} = 1.2 \times 10^{-5}$ and a standard deviation of 0.2.

Two kind of computations are performed: the first one is quasi-static, where load is supplied sufficiently slowly in order to ignore inertia effects, while the second one is dynamic, obliging us to take inertia effects into account and to compute dynamic response by the time-integration scheme presented above. Although the last kind of analysis is not really suited to such a quasi-static load, we can check that the dynamic response effectively converges to the given result of the static analysis. We use the explicit scheme for the dynamic analysis, with $\Delta t = 10^{-7}$. A Rayleigh damping $C = a M + b K$ is introduced with $a = 0.5$ and $b = 10^{-4}$. Both analysis are performed on a Voronoi cell heterogeneous structure representation of $20 \times 20$ particles. We show in Fig. 8 the response of the specimen for the quasi-static loading, in terms of stress $\sigma$ versus strain $\varepsilon$.

The second parametric study in this example concerns the effect of applied loading rate, where dynamics modification of crack pattern becomes more and more pronounced (see Fig. 9): the main crack initiates from the right side, and propagates more or less smoothly depending upon the loading rate. This effect can be easily explained: for the quasi-static load, the main crack progresses through by connecting the jagged profile of

![Fig. 8. Response for traction test with a quasi-static resolution.](image-url)
the successive weakest links. If the loading rate increases, inertial effects appear to even out the random distribution of the successive weakest links and the main crack propagates in a more smooth manner with a dominant direction perpendicular to the loading axis.

5.2. Beam spaling

We study in this example the spaling of a beam in a three-point bending test loaded by an impulse-type loading at the upper face in the middle of the span (see Fig. 10a). This kind of experimental procedure for dynamic fracture has been developed by Eibl et al. [12]. The chosen beam is 2-meter long, with a depth of 0.3 m. Material parameters are the same as in the previous study with $E = 30$ GPa, $G = 15$ GPa, $\rho = 23$ kN/m, $\epsilon^c = 1.5 \times 10^{-4}$ and $\theta^c = 1.2 \times 10^{-5}$ each one with the standard deviation of 0.2. The Voronoi cell representation of the heterogeneous structure employs $60 \times 10$ cells. The boundary conditions for the model are chosen as follows: both displacement of the lower left corner are fixed, and only the vertical displacement of the lower right corner is fixed. The loading is applied in the form of a rectangular pulse in the mid-span of the beam as shown in Fig. 10. This kind of short duration pulse loading leads to the creation of a compressive wave propagating through the specimen, which eventually reflects on the lower face of the beam. Thereupon the compressive wave turns into a traction wave, creating the cracks in that region of the beam, which when completely connected would lead to spaling at the lower face.

It is hard to imagine that any continuous model could provide an accurate enough description of this kind of experiment where multi-fragmentation plays an important role (see Fig. 11 with numerical results from [23] obtained by a viscoplasticity model available in DYNAMES).

On the other hand, we show in Fig. 12 that the proposed discrete approach employing the beam lattice network is capable of reproducing the main features of

Fig. 9. Breaking pattern for the quasi-static analysis (left) and for the dynamic one at a loading rate of $2 \times 10^{-2}$ m/s (middle).

Fig. 10. (a) Beam geometry and (b) time evolution of the loading.

Fig. 11. Computation of the beam spaling with a continuous description.

Fig. 12. Crack patterns at $t = 350$ µs, $t = 500$ µs and $t = 1000$ µs. Displacements are amplified by a factor 400.
the behavior observed in the test. The first propagating wave reaches the lower face at $t = 90 \mu s$ producing the link rupture upon the wave reflection and the spaling with aggregate of particles being ejected down, just like observed experimentally.

6. Conclusion

We have presented in this work the discrete model of heterogeneous structures based on a Voronoi cell representation, which can be used for modeling of brittle fracture phenomena for both quasi-static and dynamic loading. The first level of application for this kind of model development concerns the microscale, where each cell would roughly correspond to a typical grain size. The examples presented herein deal with the grain size typical of concrete structures of the order of millimeter to centimeter. The heterogeneous structures representation by Voronoi cells can be reconstructed analytically or simply obtained by scanning the real specimen. Between the two kinds of forces which govern the global response, the cohesive forces will influence both elastic response and initial rupture patterns, whereas the contact forces between interacting detached particles will govern the fragmentation process. One of the main originalities of the proposed model with respect to the available models of this kind concerns the use of geometrically nonlinear beam to model cohesive links, which allows to handle properly the (local) large rotations in a heavily damaged (or fractured) zone as well as a reliable representation of the fragmentation process where cohesive forces remain acting in a group of particles which split from the main structure.

The second level of application for this kind of model concerns the mesoscale, which is a viable option for making it applicable to modeling of complex structures. Quite an appealing feature of the proposed model with respect to the generalization to mesoscale relates to the fact that the latter can be completely accomplished within the framework of the presented beam theory, accounting for an equivalent form of multi-layer section properties, and replacing the random material properties by deterministic average values. One could thus develop a micro/meso approach which would be more suitable for handling a heterogeneous stress states and a combined representation with different levels of refinement (micro vs. meso) of different zones of a complex structure. In such a case one would need to develop as well the solution strategies suitable for a two-scale approach of this kind (e.g. [19]).

In order to ensure result reliability for dynamic fracture a special care should be taken of numerical integration scheme and the choice of damping which should control the spurious participation of high frequency noise in the computed response. The recent work on time-stepping schemes for stiff equations damping for geometrically nonlinear beam model in [16] might prove quite pertinent in that sense.

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References


