Study of structural assemblies in the presence of uncertainties

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Abstract   The objective of the work presented is to develop a suitable strategy for the sensitivity analysis of assemblies including contact with friction, gaps, elastically bonded joints… The approach consists in developing an application of the LArge Time INcrement (LATIN) [1] method for the parametric analysis of static problems with multiple joints. Our approach is based on a decomposition of the assembly into substructures and interfaces. The interfaces play the vital role of enabling local nonlinearities, such as contact and friction, to be modeled easily and accurately. The problem is solved in each substructure by the finite element method, and an iterative scheme based on the LATIN method is used for the global resolution.

Key words: assemblies, contact, uncertainties, probabilistic mechanics, polynomial chaos

INTRODUCTION

The solutions to deterministic problems are often calculated by finite element analysis (FEA). For structural engineers, the incorporation of parametric uncertainties of a system into a mechanical model represents a challenge; however, without this information, the structural response could not be analyzed accurately, particularly in terms of reliability. These parametric system uncertainties may affect the mechanical and geometric properties, the boundary conditions… In the case of structural assemblies, one’s knowledge of the friction coefficients or of the stiffness of bonded joints is especially poor. In order to take such uncertainty into account, it is necessary to calculate the response of the structure for all possible sets of values of the design parameters or to use a probabilistic structural analysis approach [2].

Our approach is based on a decomposition of the assembly into substructures and interfaces. The interfaces play the vital role of enabling local nonlinearities, such as contact and friction, to be modeled easily and accurately. The problem is solved in each substructure by the finite element method, and an iterative scheme based on the LATIN method is used for the global resolution.

The first approach proposed consists in calculating response surfaces such that each point of a surface is associated with a design configuration. Each design configuration corresponds to a set of values of all the variable parameters (friction coefficients, prestress) which are introduced into the mechanical analysis. A full calculation is needed for each point. Here, as an alternative to carrying out these full calculations, we propose to take advantage of the capability of the LATIN method to reuse the solution of a given problem (for one set of parameters) to solve similar problems (for the other sets of parameters) [3]. The numerical examples presented in the paper concern 3D assemblies. For some of these examples, over a thousand different calculations had to be carried out for the parametric study. The comparison of the computation costs with those of classical industrial codes shows the algorithm is very efficient when applied to the case of multiple resolutions for the analysis of static problems with contact and friction.
The second technique proposed addresses the random response of assemblies whose interfaces have uncertain characteristics. Thus, the randomness of the response comes from the random behavior of the connections. In this context, the random response is developed using a Polynomial Chaos Expansion (PCE) [4] coupled with the LATIN approach. A dedicated computational strategy to determine the random response of assemblies with probabilistic interface characteristics is presented [5]. The numerical example concerns an assembly of structures connected by adhesively bonded joints with random stiffness characteristics. The results are compared with those of the Monte Carlo method and the stochastic finite element method. Since the uncertain parameters are present only in the joints, the use of the LATIN method results in a drastic reduction of the computation costs.

The LATIN method

Here, we will review only the main aspects of the LATIN method. The details of the method itself can be found in [1] and those of its particular application to computational contact problems in [6, 7]. An assembly is composed of a set of substructures (each substructure is a component of the assembly) which communicate with one another through interfaces (each interface represents a connection), see Fig. 1. Each interface is a mechanical entity with its own variables and its specific behavior which depends on the type of connection. Many different connection types can be modeled by this approach, but in this paper we consider only classical contact connections. Two connected substructures are denoted \( \Omega^E \) and \( \Omega^{E'} \) and the associated interface is designated by \( \Gamma^{EE'} \). The interface variables are two force fields \( \mathbf{f}^E \) and \( \mathbf{f}^{E'} \) and two dual velocity fields \( \mathbf{w}^E \) and \( \mathbf{w}^{E'} \) (Fig. 1). By convention, \( \mathbf{f}^E \) and \( \mathbf{f}^{E'} \) are the actions of the interface on the substructures and \( \mathbf{w}^E \) and \( \mathbf{w}^{E'} \) are the velocities of the substructures seen from the interface.

![Substructures and interface](image)

The displacement field at any point \( M \) of \( \Omega^E \) and at any time \( t \) of \( [0, T] \) is \( u^E(M, t) \); the associated space is \( \mathcal{U}^{[0,T]} \). \( \mathbf{e} \) is the strain field and the current state of the structure is characterized by the stress field \( \sigma^E \). The mechanical problem to be solved in each substructure is:

1. Kinematic admissibility:
   \[
   \mathbf{e} = e(u^E); \quad u^E(M, t) \big|_{\partial \Omega^E} = \mathbf{w}^E(M, t); \quad u^E \in \mathcal{U}^{[0,T]}
   \]  

2. Equilibrium: \( \forall \mathbf{u}^E \in \mathcal{U}^{[0,T]} \)
   \[
   \int_{\Omega^E} \nabla \cdot (\sigma^E \mathbf{e}(\mathbf{u}^E)) \, d\Omega^E - \int_{\Omega^E} \mathbf{f} \cdot \mathbf{u}^E \, d\Omega^E - \int_{\Gamma^{EE'}} \mathbf{f}^{E'} \cdot \mathbf{w}^{E'} \, dS = 0
   \]  

3. Elastic behavior: \( \forall M \in \Omega^E \) and \( \forall t \in [0, T] \)
   \[
   \sigma^E(M, t) = D \mathbf{e}(u^E(M, t))
   \]

where \( D \) is Hooke’s operator.

The mechanical problem to be solved on each interface is:
Find the histories of the force fields \( f^E(M,t) \) and \( f^E'(M,t) \) and of the velocity fields \( \dot{w}^E(M,t) \) and \( \dot{w}^E(M,t) \) such that:

- Equilibrium: \( \forall \ M \in \Gamma^{EE'} \) and \( \forall \ t \in [0, T] \)

\[
f^E(M,t) + f^E'(M,t) = 0 \tag{4}
\]

- Behavior: \( \forall \ M \in \Gamma^{EE'} \) and \( \forall \ t \in [0, T] \)

\[
f^E(M,t) = R(\dot{w}^{EE'}(M,t), \tau \in [0, t]) \tag{5}
\]

where the behavior is expressed as a nonlinear evolution law \( R \) between the forces and the rate \( \dot{w}^{EE'} \) of jump in displacement across the interface which is defined by:

\[
\dot{w}^{EE'} = \dot{w}^E - \dot{w}^E
\tag{6}
\]

For example, a perfect connection between two substructures would be modeled by the following behavior:

\[
\dot{w}^{EE'} = 0 \tag{7}
\]

The form of the evolution law \( R \) in the case of frictional contact conditions is described in [3].

A LATIN (LArge Time INcrement) approach [1] is used to solve the problem. The solution \( s \) of the problem is written as a set of time-dependent fields on each substructure and related interfaces:

\[
s = \sum_{E} s^E \ ; \ s^E = \{ u^E(M,t), \sigma^E(M,t), \dot{w}^E(M,t), \dot{f}^E(M,t) \} \ t \in [0, T]
\tag{8}
\]

The LATIN approach is based on the idea of isolating the difficulties in order not to have to solve a global problem and a nonlinear problem at the same time. The equations are split into two groups with the following two sets of solutions:
- the set \( A_d \) of solutions \( s^E \) to the linear equations related to the substructures (Eq. 1 to 3)
- the set \( \Gamma \) of solutions \( s^E \) to the local equations (which may be nonlinear) related to the interfaces (Eq. 4 to 5)

The search for the overall solution (i.e. the intersection of the two sets) is conducted iteratively by constructing approximate solutions \( s \) which verify the two groups of equations alternatively on the complete time history. Thus, each iteration in the process is composed of two stages:

**Local stage:** for \( s_n \in A_d \), find \( \hat{s} \) such that:

\[
\hat{s} \in \Gamma \quad (interfaces) \tag{9}
\]

\[
\hat{s} - s_n \in E^+ \quad (search \ direction) \tag{10}
\]

**Global stage:** for \( \hat{s} \in \Gamma \), find \( s_n \) such that:

\[
s_{n+1} \in A_d \quad (substructures) \tag{11}
\]

\[
s_{n+1} - \hat{s} \in E^- \quad (search \ direction) \tag{12}
\]

The search directions are chosen such that the convergence of the algorithm is ensured [1].

**FIRST APPROACH PROPOSED**

The first approach proposed consists in calculating response surfaces such that each point of a surface is associated with a design configuration. Each design configuration corresponds to a set of values of all the variable parameters (friction coefficients, prestress) which are introduced into the mechanical
analysis. At each iteration, the LATIN method leads to an approximate solution to the problem over the whole time interval. Therefore, the trick is to reuse this approximation (associated to one set of values of all the design parameters) to find the solution to another design configuration (another set of the design parameters) similar to the one for which it was calculated in the first place. Our multiple solution method uses the fact that the LATIN algorithm can be initialized with any solution (usually an elastic solution) provided that it verifies the admissibility conditions. Therefore, the key to our technique is to initialize the process associated with a new design configuration using the results of the calculation carried out on the first set of values of the design parameter. In this manner, a first approximation of the solution to the new design with a strong mechanical content is immediately available from the start.

In this particular case of elastic structures in contact, the interfaces play a vital role: they enable one to initiate the calculation on the new design configuration without having to save all data on the substructures as well as to search for the solution of the new design configuration with an initial solution well-suited to the target problem. In the best-case scenario, only a few iterations are necessary: the solution to the problem is obtained at low cost. If the solutions to the design configurations are close enough, the latter can still be derived at a significantly lower cost than by using a full calculation.

For the parametric study presented herein, we just change the parameters between iterations. Thus, the new computation is initialized by the solution to the previous one. If the parameters change slowly, the two solutions are close and only a few iterations are needed to reach convergence in the new calculation.

**EXAMPLES**

1. First academic example Let us consider the simple example of a cube subjected to frictional contact against a rigid body. This 40x40x40mm cube is subjected to friction on plane (z = 0). Symmetry conditions are prescribed at planes (x = 0) and (y = 0). The cube is pressed against the rigid body through the application of a pressure $P_1 = 5$ MPa on its upper side. The other two sides are successively loaded and unloaded ($P_2 = 10$ MPa and $P_3 = 10$ MPa) in order to generate friction on the contact surface. The geometry and loading conditions are detailed in Fig. 2. We consider the quasi-static, small-perturbation problem with an elastic material (Young’s modulus $E = 130000$ MPa and Poisson’s coefficient $\nu = 0.2$). The mesh is composed of 100 eight-node brick elements (460 degrees of freedom).

![Fig. 2 Cube example, the model and loading conditions](image)

Fig. 3 shows the deformed shape of the cube (magnified 400 times) for a friction coefficient $\mu = 0.4$ after completion of the five loading stages (application of $P_1$, application of $P_2$, application of $P_3$, removal of $P_2$ and removal of $P_3$). Each loading stage was completed in 10 time steps. The problem’s parameter is the friction coefficient $\mu$, which varies from 0 to 2 in increments of 0.1. We studied the displacements of Point A (Fig. 2) in the contact plane ($z = 0$). Fig. 4 shows the paths followed by Point A for the 21 values of $\mu$ and the evolution of the error indicator which increases every time the friction coefficient changes.
were very close (as shown by Fig. 4). Thus, even though the first calculation cost three times as much as the direct calculation, the computational cost was compared with the cost of a direct approach using ABAQUS. Fig. 5 shows the cost of each of the 21 computations. One can observe that in the direct calculations using ABAQUS the computational cost increased with the friction coefficient. In the parametric study using the LATIN method, the computational cost decreased because the results of the successive computations were very close (as shown by Fig. 4). Thus, even though the first calculation cost three times as much as the direct method, the global cost of the parametric study using the LATIN method was lower by a factor of 1.5.

2. 3D Assembly A bolted connection is considered, the dimensions of the studied part of the connection are presented on Fig. 6. The connection between three plates is assumed by two prestressed bolts. The bolts and the plates are composed of the same material (Young’s modulus $E = 200000$ MPa and Poisson’s coefficient $\nu = 0.3$). The prestress of the bolts is assumed by a relative displacement between the body and the head of each bolt. The two relative displacements are denoted $g_1$ and $g_2$ (Fig. 6) and the friction coefficient is denoted $\mu$. In the parametric study, we study the influence of the prestress of the bolts and of friction on the transmission of forces. The same Coulomb friction coefficient is used on each
contact zones. There is thus three parameters: one friction coefficient and one prestress in each bolt. The friction coefficient can take 9 different values (0.1 to 0.5, step 0.05). The pre-stresses can take 12 different values (0.05 mm to 0.3 mm, step 0.025 mm). For the complete parametric study 1296 computations have thus to be performed.

![Diagram](image)

**Fig. 6 Scheme of the structures and dimensions – thickness: 40 mm**

The same mesh have been used for all the computations. It is presented on Fig. 7. It is composed of 10 705 linear elements (pyramids or bricks) and 8 090 nodes. The total number of degrees of freedom is then 24 270. This number does not included the possible additional contact variables (Lagrange multipliers…). The computation is carried out on two steps:

- step 1: pre-stress of the bolts (duration 1s - 1 time increment asked).
- step 2: application of the load (duration 1s - 10 time increments asked).

In order to estimate the capabilities of the LATIN method for the treatment of frictional contact problems a comparison with the industrial finite element code ABAQUS have been carried out on one parametric configuration ($g_1 = 0.05$ mm, $g_2 = 0.025$ mm and $\mu = 0.3$). Fig. 7 presents the comparison of the global response of the connection (displacement of one point of the loaded surface). The ABAQUS and the LATIN solutions are very closed.

![Graph](image)

**Fig. 7 Mesh of the assembly and Displacement of one point of the loaded surface**

For convergence reasons ABAQUS solver ran more time increments than asked (10 for step 1 and 62 for step 2). The results of the comparison are presented in Table 1.

<table>
<thead>
<tr>
<th></th>
<th>ABAQUS</th>
<th>LATIN Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time steps</td>
<td>72</td>
<td>11</td>
</tr>
<tr>
<td>Slip tolerance</td>
<td>$10^{-4}$</td>
<td>0</td>
</tr>
<tr>
<td>CPU Time (mn)</td>
<td>374</td>
<td>38.1</td>
</tr>
<tr>
<td>Wall Clock Time (mn)</td>
<td>407</td>
<td>40</td>
</tr>
</tbody>
</table>

One can notice that on this single computation the LATIN method is 10 times more efficient than a classical Finite Element code. This efficiency, in terms of size of the problem and in term of computational time, as already been hown and discussed in [7].
Table 2. Computational costs - 1296 computations

<table>
<thead>
<tr>
<th>Wall Clock Time</th>
<th>h</th>
<th>days</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABAQUS direct</td>
<td>8 791</td>
<td>366</td>
</tr>
<tr>
<td>LATIN direct</td>
<td>864</td>
<td>36</td>
</tr>
<tr>
<td>LATIN multiple solution</td>
<td>168</td>
<td>7</td>
</tr>
</tbody>
</table>

Fig. 8 summarizes all the realized calculations. The evolution of the maximum transmission force is plotted for each value of the friction coefficient $\mu$ as a function of the two prestress of bolt $g_1$ and $g_2$. One can notice that the force varies slowly according to the prestress, but strongly according to the friction coefficient.

![Fig. 8 Summarized presentation of all the results](image)

SECOND APPROACH PROPOSED

Here, we are dealing with structural assembly problems in which the behavior of the components and the external loads are deterministic: the randomness of the response comes from the random behavior of the connections. The system’s parameters, calibrated using experimental data, are modeled as random variables or processes which are assumed to be properly represented by a set of random variables $\{\xi_i(\theta)\}$, where $\theta$ belongs to the space of random events $\Omega$. The state of the system, again modeled as a random variable or process, resides in a certain Hilbert space. A set of basis functions $\{\Psi_i\}$ will also be identified in this space. Considering static loading cases, the main differences from the initial problem are the interfaces equations:

- Equilibrium: $\forall \ M \in \Gamma^{EE}$, $\forall \ \theta \in \Omega$

  \[ \mathbf{f}^E(M, \theta) + \mathbf{f}^E(M, \theta) = 0 \]  
  \[ (13) \]

- Behavior: $\forall \ M \in \Gamma^{EE}$, $\forall \ \theta \in \Omega$

  \[ \mathbf{f}^E(M, \theta) = k(\alpha(\theta))\mathbf{w}^{EE}(M, \theta) \]  
  \[ (14) \]

where $k$ is the interface stiffness operator, which depends on a random material parameter $\alpha(\theta)$. 
Using a standard finite element discretization for the displacement field within the substructures and at the interfaces:

\[ u^E(M, \theta) = N U^E(\theta) \quad \text{and} \quad \varepsilon(u^E) = B U^E(\theta) \quad (15) \]

where \( U^E \) is the vector of nodal displacements and \( N \) is the vector of classical finite element basis functions. The nodal variables \( U^E(\theta) \) can be formally expressed as a nonlinear functional of the set \( \{ \xi(\theta) \} \) used to represent the material stochastic property. It has been shown [8] that this functional dependence can be expanded in terms of polynomial chaoses. Then, the truncated PCE of the response takes the form:

\[ U^E(\theta) = \sum_{i=1}^{P} U^E_i \psi_i(\theta) \quad (16) \]

where \( \{ \psi_i(\theta) \} \) are polynomials in the Gaussian random variables \( \{ \xi(\theta) \} \). These can be shown to form a complete basis of the Hilbert space of second-order random variables. The number of polynomials \( P \) depends on the order \( p \) of the PCE and on the number of stochastic parameters. Let us assume that the material parameters are constant along an interface. Should this not be the case, a Karhunen-Loève expansion could easily be used to represent the spatial randomness of the interface characteristics [9].

The stochastic material property \( \alpha(\theta) \) is represented by:

\[ \alpha(\theta) = \bar{\alpha}(1 + \delta \xi(\theta)) \quad (17) \]

where \( \bar{\alpha} \) is the mathematical expectation of \( \alpha(\theta) \), \( \delta \) the coefficient of variation (standard deviation versus expectation) and \( \xi(\theta) \) the standard normal random variable: \( \xi(\theta) \sim N(0,1) \). For non-Gaussian material properties, the PCE is used to represent the material property: the case of a Gaussian process is a particular case chosen for simplicity’s sake [10].

Using this decomposition presented, the resolution of the two stages of the LATIN method can be summarized as follows:

- Resolution for the local stage: we assumed the stiffness to be constant at the interface. Therefore, the solution is achieved through the resolution of small independent systems (size \( (P + 1) \)) at each node.
- Resolution for the global stage: this consists of solving \( (P + 1) \) independent linear systems for each substructure. It is important to note that the matrices which appear in the system remain constant during the iterations and, therefore, need to be factorized only once before the first iteration. An even more important observation is that the problems on the substructures are completely independent of one another and could be solved in parallel very efficiently.

**EXAMPLE**

We consider an assembly of three parts (Fig. 9) connected by two adhesively bonded joints. The dimensions are expressed in millimeters and the material properties are shown in Table 3. 2D and 3D cases has been considered. Substructure (3) is subjected to a vertical force \( F = 250 \) N. The quantities of interest are the displacement of point A at the top of substructure (3) and the maximum stresses in the bonded joints. The adhesively bonded joints (denoted I and II on Fig. 9) are modeled with interfaces. An interface’s normal \( (k_n) \) and tangential \( (k_t) \) stiffness are derived from the adhesive’s characteristics:

\[ k_n = \frac{E}{e} \quad k_t = \frac{E}{2(1 + \nu)} \quad (18) \]

where \( E \) is Young’s modulus, \( \nu \) Poisson’s coefficient and \( e \) the thickness of the adhesive. In this case, Young’s modulus is a random parameter:

\[ E(\theta) = \bar{E}(1 + \delta \xi(\theta)) \quad (19) \]
Table 3. Material properties of the assembly

<table>
<thead>
<tr>
<th>Substructure</th>
<th>$E$ (GPa)</th>
<th>$v$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Base (1)</td>
<td>120</td>
<td>0.3</td>
</tr>
<tr>
<td>L-shaped (2)</td>
<td>200</td>
<td>0.3</td>
</tr>
<tr>
<td>Vertical part (3)</td>
<td>70</td>
<td>0.3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Interface</th>
<th>$E$ (MPa)</th>
<th>$v$</th>
<th>$e$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(I)</td>
<td>500</td>
<td>0.45</td>
<td>0.3</td>
</tr>
<tr>
<td>(II)</td>
<td>1000</td>
<td>0.45</td>
<td>0.3</td>
</tr>
</tbody>
</table>

Fig. 9 Assembly containing two bonded joints and 3D mesh used (37174 dof in 2D, 188 016 dof in 3D)

Fig. 10 and Table 4 show the comparison of the probability density function of the displacement $u_x$ of point A obtained by the LATIN method with that obtained by a classical Monte Carlo method (10,000 draws). The results are very close. The LATIN results are the same as those obtained by direct resolution.

Table 4. Comparison of the CPU time for the calculations in 2D and 3D

<table>
<thead>
<tr>
<th>Calculation</th>
<th>CPU Time (s) – 2D</th>
<th>CPU Time (h) – 3D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monte Carlo (10 000 draws)</td>
<td>75 700</td>
<td>-</td>
</tr>
<tr>
<td>Polynomial Expansion - Direct</td>
<td>780</td>
<td>-</td>
</tr>
<tr>
<td>Polynomial Expansion - LATIN</td>
<td>185</td>
<td>3h55 (parallel computing)</td>
</tr>
</tbody>
</table>
REFERENCES


CONCLUSIONS

The two proposed approaches based on the LATIN method can be very efficient numerically. The first strategy is based on its capability to reuse the solution to a given problem in order to solve similar problems. Numerical examples showed the very good behavior of the algorithm applied to the case of multiple resolutions in the analysis of 3D assemblies. The solution to the initial problem is a very good starting point for the calculations conducted on other problems provided that these calculations do not exert excessive perturbations on the response. Moreover, the interfaces play a vital role in allowing a considerable reduction of the computation costs. In the second approach, the uncoupled treatment of the local and global problems leads to a considerable reduction of problemsizes. Another important point is that the linear systems corresponding to the substructures are independent of one another and could be solved in parallel very efficiently. This approach is quite general by nature and should be applicable to a number of other nonlinear problems.